Analytic Regularity and GPC Approximation for Control Problems Constrained by Linear Parametric Elliptic and Parabolic PDEs

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ANALYTIC REGULARITY AND GPC APPROXIMATION FOR CONTROL PROBLEMS CONSTRAINED BY LINEAR PARAMETRIC ELLIPTIC AND PARABOLIC PDES*

ANGELA KUNOTH† AND CHRISTOPH SCHWAB‡

Abstract. This paper deals with linear-quadratic optimal control problems constrained by a parametric or stochastic elliptic or parabolic PDE. We address the (difficult) case that the state equation depends on a countable number of parameters i.e., on σ_j with $j \in \mathbb{N}$, and that the PDE operator may depend non-affinely on the parameters. We consider tracking-type functionals and distributed as well as boundary controls. Building on recent results in [CDS1, CDS2], we show that the state and the control are analytic as functions depending on these parameters σ_j. We establish *sparsity of generalized polynomial chaos (gpc)* expansions of both, state and control, in terms of the stochastic coordinate sequence $\sigma = (\sigma_j)_{j\geq 1}$ of the random inputs, and prove convergence rates of best N-term truncations of these expansions. Such truncations are the key for subsequent computations since they do *not* assume that the stochastic input data has a finite expansion. In the follow-up paper [KS], we explain two methods how such best N-term truncations can practically be computed, by greedy-type algorithms as in [SG, G], or by multilevel Monte-Carlo methods as in [KSS]. The sparsity result allows in conjunction with adaptive wavelet Galerkin schemes for sparse, adaptive tensor discretizations of control problems constrained by linear elliptic and parabolic PDEs developed in [DK, GK, K], see [KS].

Key words. Linear-quadratic optimal control, linear parametric or stochastic PDE, distributed or boundary control, elliptic or parabolic PDE, analyticity, generalized polynomial chaos approximation.

AMS subject classifications. 41A, 65K10, 65N99, 49N10, 65C30.

1. Introduction.

Increasingly, the simulation and design of complex systems requires the numerical solution of partial differential equations (PDEs) involving a large number of parameters, thereby leading to PDEs on high dimensional, so-called design spaces. Typical examples are PDEs with stochastic coefficients or stochastic PDEs driven by noise which are transformed into deterministic high-dimensional PDEs by means of spectral approaches like Wiener chaos or Karhunen-Loève expansions [GS, KX, Sch, W]. Such approaches for quantification of uncertainty pose enormous challenges for numerical simulations already for a single elliptic PDE and has triggered several contributions over the past years, see, e.g., [BNT, BTZ, CDS1, HLM, SG, ST].

Of particular recent interest are optimal control problems of uncertain systems governed by linear parametric or stochastic PDEs. In PDE-constrained control with a tracking-type optimization functional, the goal is to steer the solution y of the PDE,

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the *state*, towards a prescribed desired state, the *target* y_*, in a least-squares sense while minimizing the effort for the *control* u. If the governing PDE depends on a (possibly *infinite*) *sequence* of parameters arising, for example, from random field inputs in models of uncertainty as diffusion coefficients, this requires the solution of the control problem for *each instance* of the parameters. Already for a single random variable σ in the diffusion coefficient, the computational expense of sampling state and control is enormous: e.g., in a Monte-Carlo simulation with N independent draws, each realization of this variable requires the solution of the whole control problem. Necessarily, this results in N solutions of the control problem to be performed. Although easy to realize and therefore quite popular, Monte-Carlo methods have the drawback that the slow approximation rate of $N^{-1/2}$ requires many samples to reach a sufficiently low error, and this convergence rate cannot be improved even for smooth solutions. Moreover, the case of *infinitely* many parameters cannot be handled immediately by Monte-Carlo methods. As an alternative to these schemes, spectral approaches based on so-called *Wiener* or *generalized polynomial chaos (gpc) expansions* in terms of orthogonal polynomials have been introduced in [W], see also [KX, Sch]. These lead to deterministic parametric PDE-constrained control problems which depend on possibly infinitely many parameters.

Already for deterministic, non-parametric linear-quadratic control problems constrained by elliptic state equations, one needs to solve as first order necessary and sufficient conditions for optimality a coupled system of linear PDEs for the state y and the *adjoint state* p each, and a third equation coupling p with the control u. For such systems of PDEs, if the solutions are smooth, best approximations are obtained with discretizations on uniform grids. In this case, multilevel solvers can produce the solution triple in optimal linear complexity, see, e.g., [BoKu, BoSch, EG, Ha, SSZ]. In the case of nonsmooth solutions, for such systems of coupled PDEs, in recent years solvers became available which produce *optimal numerical approximations* of the solution triple (y, p, u) in the following sense. Accuracy versus work to obtain these approximations is *provably proportional* to those of (wavelet-)best M-term approximations of the solution triple (y, p, u). This means that the complexity of the scheme is as good as if one knew the solutions beforehand, expanded them in a wavelet representation and selected the M largest coefficients, yielding best M-term approximations. Thus, these schemes allow to achieve arbitrary, user-specified accuracy ε with an order of arithmetic operations that is the best possible one afforded by the regularity of the data. These solvers are based on adaptive wavelet schemes for which convergence and optimal complexity have been established for distributed and Neumann boundary control problems in [DK] and for Dirichlet control problems in [K]. The related concepts from nonlinear approximation theory which are used to establish these results can be found in [DeVK] and the articles therein.

When the control problem is constrained by a (deterministic) *parabolic evolution* PDE, the necessary conditions for optimality lead to a system of parabolic PDEs coupled *globally* in time. Traditional time-stepping methods for the primal (forward in time) problem require to store the solution of the adjoint (backward in time) problem at all times, and vice versa. These enormous storage demands led to the development of special techniques like checkpointing, see, e.g., [GW]. To overcome the limitations posed by time-stepping approaches, we followed in [GK] the idea of [DL, SS] to employ a *full weak space-time formulation* of the evolution PDE constraint. For the resulting system of coupled operator equations, we developed a wavelet-based time-space adaptive algorithm and proved *convergence* and *optimal complexity estimates* in the sense explained above. We wish to point out that convergence results for adaptive finite elements (or any other adaptive Galerkin approximation method) for such control problems do not yet exist, not even for the case of control problems constrained by elliptic PDEs. Nevertheless, adaptive finite element methods seem to work well practically, see [MV] for corresponding space-time methods for parabolic optimization problems.

The purpose of the present paper is to provide a *provably convergent* and *optimally efficient* adaptive algorithm for solving a large class of linear-quadratic optimal control problems constrained by elliptic or parabolic PDEs with *stochastic or possibly infinitely many countable parametric* coefficients and different types of control. The solution triple of the necessary and sufficient conditions for optimality (y, p, u) will then depend on time t, space $x \in \mathbb{R}^{\mathfrak{d}}$ and on the parameter sequence $\sigma \in [-1, 1]^{\mathbb{N}}$.

In this paper, we prove sparsity results for the dependence of the solution triple (y, p, u) on the parameter $\sigma = (\sigma_j)_{j\geq 1} \in [-1, 1]^{\mathbb{N}}$. Therefore, this approach may be viewed as a "semi-discretization" and the derivation of a-priori error estimates with respect to σ. This will be achieved as follows. Building on recent results in [CDS1, CDS2], we show that the state and the control are *analytic* as functions of the parameter sequence σ. This, in turn, allows expansions of state and control in terms of tensorized Legendre polynomials of the parameter sequence, so-called *generalized polynomial chaos (gpc) expansions.* We will show that corresponding best N-term truncations of these expansions decay at certain convergence rates which are determined by sparsity properties of the random field input for the state equation. Most interesting, however, are situations where the best N-term approximation rate is higher than the rate $1/2$ achieved by Monte-Carlo schemes.

Such best N-term truncations are the key for subsequent computations since they do *not* assume that the stochastic input data has a finite expansion. This allows for random field input data (such as stochastic coefficients) and is a substantial difference to previous studies. In [BNT, BTZ] concerning elliptic PDEs, the stochastic coefficients and input data were assumed to depend on a finite number of random variables (so-called "finite-dimensional noise assumption"). Likewise, in [GLL, HLM] dealing with control problems constrained by elliptic PDEs with stochastic coefficients, the stochastic model was approximated by a deterministic one using a Karhunen-Loève expansion with a finite number of parameters. Our results allows to deal with random field input, i.e., with stochastic coefficients whose Karhunen-Loève expansions are not necessarily finite.

In order to cover a wide range of control problems, approximation results will first be derived for an abstract, parametric saddle point problem, generalizing [CDS1, CDS2]. Later, this is specified to different situations of linear-quadratic control problems with different types of PDEs and controls. The class of linear-quadratic optimal control problems we are able to handle allows for elliptic and parabolic PDEs as constraints and different type of controls (distributed, Neumann boundary and Dirichlet boundary control) and choices of norms in the optimization functional as long as the resulting saddle point system of necessary conditions (for y, p) can be proved to possess a unique solution for each choice of parameters σ.

The theoretical sparsity and N-term approximation results established in the present paper is the foundation of *sparse, adaptive tensor approximation algorithms* which will be presented in [KS], resulting in a compressive, fully discrete scheme in terms of σ, t, x.

This paper is structured as follows. In the next section, it is proved that the solution of a linear operator equation involving a general parameter-dependent saddle point operator in an abstract setting is analytic, with precise bounds on the growth of the partial derivatives. This allows us in Section 2.4 to obtain rates of N-term generalized polynomial chaos approximations. These results are specified in Section 3 to linear-quadratic control problems constrained by an elliptic PDE with distributed, Neumann or Dirichlet boundary control and in Section 4 to control problems constrained by linear parabolic PDEs. We conclude in Section 5 with some remarks how to realize this practically and how to combine the gpc approximations with discretizations with respect to space and time.

2. Parametric saddle point problems. We generalize the results of [CDS1] and study well-posedness, regularity and polynomial approximation of solutions for a family of abstract parametric saddle point problems. Particular attention is paid to the case of *an infinite sequence of* parameters. The abstract results in the present section are more general than what is required in our ensuing treatment of optimal control problems and are of independent interest. We discuss in detail in the following sections their application to optimal control problems for systems constrained by elliptic and parabolic PDEs with random coefficients.

2.1. An abstract result. Throughout, we denote by $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ two reflexive Banach spaces over $\mathbb{R}$ (all results will hold with the obvious modifications also for spaces over $\mathbb{C}$) with (topological) duals $\boldsymbol{\mathcal{X}}'$ and $\boldsymbol{\mathcal{Y}}'$, respectively. By $\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}')$, we denote the set of bounded linear operators $G : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}'$. The Riesz representation theorem associates with each $G \in \mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}')$ a unique bilinear form $\mathcal{G}(\cdot,\cdot) : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} \to \mathbb{R}$ by means of

$$\mathcal{G}(v,w) = \langle w, Gv\rangle_{\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Y}}'} \quad \text{for all } v \in \boldsymbol{\mathcal{X}}, w \in \boldsymbol{\mathcal{Y}}. \tag{2.1}$$

Here and in what follows, we indicate spaces in duality pairings $\langle\cdot,\cdot\rangle$ by subscripts.

We shall be interested in the solution of linear operator equations $Gq = g$ and make use of the following solvability result which is a straightforward consequence of the closed graph theorem, see, e.g., [BF].

Proposition 1. *An operator $G \in \mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}')$ is boundedly invertible if and only if its associated bilinear form satisfies the* inf-sup conditions*: there exists a constant $\gamma > 0$ such that*

$$\inf_{0\neq v\in\boldsymbol{\mathcal{X}}} \sup_{0\neq w\in\boldsymbol{\mathcal{Y}}} \frac{\mathcal{G}(v,w)}{\|v\|_{\boldsymbol{\mathcal{X}}}\|w\|_{\boldsymbol{\mathcal{Y}}}} \geq \gamma \tag{2.2}$$

and

$$\inf_{0\neq w\in\boldsymbol{\mathcal{Y}}} \sup_{0\neq v\in\boldsymbol{\mathcal{X}}} \frac{\mathcal{G}(v,w)}{\|v\|_{\boldsymbol{\mathcal{X}}}\|w\|_{\boldsymbol{\mathcal{Y}}}} \geq \gamma\ . \tag{2.3}$$

If (2.2) *and* (2.3) *hold, then for every $g \in \boldsymbol{\mathcal{Y}}'$ the operator equation*

$$\textit{find } q \in \boldsymbol{\mathcal{X}}: \quad \mathcal{G}(q,v) = \langle g, v\rangle_{\boldsymbol{\mathcal{Y}}'\times\boldsymbol{\mathcal{Y}}} \quad \forall v \in \boldsymbol{\mathcal{Y}} \tag{2.4}$$

admits a unique solution $q \in \boldsymbol{\mathcal{X}}$. There holds the a-priori estimate

$$\|q\|_{\boldsymbol{\mathcal{X}}} \leq \frac{\|g\|_{\boldsymbol{\mathcal{Y}}'}}{\gamma}\ . \tag{2.5}$$

2.2. Parametric operator families. In the present paper, we shall be interested in *parametric families of operators* G. We admit both, finite as well as countably infinite sequences σ of parameters. To this end, we denote by $\sigma := (\sigma_j)_{j\in\mathbb{S}} \in \mathcal{S}$ the set of parameters where $\mathbb{S} \subseteq \mathbb{N}$ is an at most countable index set. We assume the parameters to take values in $\mathcal{S} \subset \mathbb{R}^{\mathbb{S}}$. In particular, in the case $\mathbb{S} = \mathbb{N}$ it holds $\mathcal{S} \subseteq \mathbb{R}^{\mathbb{N}}$, i.e., each realization of σ is a sequence of real numbers. We shall consider in particular the parameter domain $\mathcal{S} = [-1,1]^{\mathbb{N}}$ which we equip with the uniform probability measure

$$\rho(\sigma) = \bigotimes_{j\geq 1} \frac{\sigma_j}{2} \ . \tag{2.6}$$

By $\mathbb{N}_0^{\mathbb{N}}$ we denote the set of all sequences of nonnegative integers, and by $\mathfrak{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\}$ the set of "finitely supported" such sequences, i.e., sequences of nonnegative integers which have only a finite number of nonzero entries. For $\nu \in \mathfrak{F}$, we denote by $\mathfrak{n} \subset \mathbb{N}$ the set of coordinates j such that $\nu_j \neq 0$, *with j repeated $\nu_j \geq 1$ many times.* Analogously, $\mathfrak{m} \subset \mathbb{N}$ denotes the supporting coordinate set for $\mu \in \mathfrak{F}$.

We consider *parametric* families of continuous, linear operators which we denote as $G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}')$. We now make precise the dependence of $G(\sigma)$ on the parameter sequence σ which is required for our regularity and approximation results: $G(\sigma)$ is required to be (real) analytic. Recall that a (real) analytic function is infinitely differentiable and coincides, in an open, nonempty neighborhood of each point, with its Taylor series at that point. This is detailed in the following

ASSUMPTION 1. *The parametric operator family $\{G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}') : \sigma \in \mathcal{S}\}$ is a* regular $\mathfrak{p}$-analytic operator family *for some $0 < \mathfrak{p} \leq 1$, i.e.,*

(i) $G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}')$ is boundedly invertible for every $\sigma \in \mathcal{S}$ with uniformly bounded inverses $G(\sigma)^{-1} \in \mathcal{L}(\boldsymbol{\mathcal{Y}}', \boldsymbol{\mathcal{X}})$, i.e., there exists $C_0 > 0$ such that

$$\sup_{\sigma\in\mathcal{S}} \|G(\sigma)^{-1}\|_{\mathcal{L}(\boldsymbol{\mathcal{Y}}', \boldsymbol{\mathcal{X}})} \leq C_0; \tag{2.7}$$

(ii) for any fixed $\sigma \in [-1,1]^{\mathbb{N}}$, the operators $G(\sigma)$ are (real) analytic functions with respect to σ. Specifically, this means that there exists a nonnegative sequence $b = (b_j)_{j\geq 1} \in \ell^{\mathfrak{p}}(\mathbb{N})$ such that

$$\forall \nu \in \mathfrak{F}\backslash\{0\} : \quad \sup_{\sigma\in\mathcal{S}} \left\| (G(0))^{-1} (\partial_\sigma^\nu G(\sigma)) \right\|_{\mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}})} \leq C_0 b^\nu \ . \tag{2.8}$$

Here $\partial_\sigma^\nu G(\sigma) := \frac{\partial^{\nu_1}}{\partial\sigma_1} \frac{\partial^{\nu_2}}{\partial\sigma_2} \cdots G(\sigma)$; the notation b^ν signifies the (finite due to $\nu \in \mathfrak{F}$) product $b_1^{\nu_1} b_2^{\nu_2} \ldots$ and we use the convention $0^0 := 1$.

Note that the estimates in (2.8) are taken for all possible derivatives from the *finite* set $\mathfrak{F}$. This will be one of the keys for dealing with infinitely many parameters σ. The regularity parameter $\mathfrak{p} \in (0,1]$ which controls the decay of the derivatives $\partial_\sigma^\nu G(\sigma)$ will play a prominent role throughout this paper. The dependence of $G(\sigma)$ on σ formulated in Assumption 1 allows for very general situations. One of the most frequently appearing cases is the following.

Affine Parameter Dependence. The special case of *affine parameter dependence* arises, for example, in diffusion problems where the diffusion coefficients are given in terms of a Karhunen-Loève expansion (see, e.g., [ST] for such Karhunen-Loève expansions and their numerical analysis, in the context of elliptic PDEs with random coefficients). Then, there exists a family $\{G_j\}_{j\geq 0} \subset \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}')$ such that $G(\sigma)$

can be written in the form

$$\forall \sigma \in \mathcal{S}: \quad G(\sigma) = G_0 + \sum_{j\geq 1} \sigma_j G_j \ . \tag{2.9}$$

We shall refer to $G_0 = G(0)$ as "nominal", or 'mean-field" operator, and to G_j, $j \geq 1$, as "fluctuation" operators. In order for the sum in (2.9) to converge, we impose the following assumptions on $\{G_j\}_{j\geq 0}$. In doing so, we associate with the operators G_j the bilinear forms $\mathcal{G}_j(\cdot,\cdot) : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} \to \mathbb{R}$.

Assumption 2. *The family of operators $\{G_j\}_{j\geq 0}$ in (2.9) satisfies the following conditions:*

1. *The "mean field" operator $G_0 \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}')$ is boundedly invertible, i.e. (cf. Proposition 1) there exists $\gamma_0 > 0$ such that*

$$\inf_{0\neq v\in\boldsymbol{\mathcal{X}}} \sup_{0\neq w\in\boldsymbol{\mathcal{Y}}} \frac{\mathcal{G}_0(v,w)}{\|v\|_{\boldsymbol{\mathcal{X}}}\|w\|_{\boldsymbol{\mathcal{Y}}}} \geq \gamma_0 \tag{2.10}$$

and that

$$\inf_{0\neq w\in\boldsymbol{\mathcal{Y}}} \sup_{0\neq v\in\boldsymbol{\mathcal{X}}} \frac{\mathcal{G}_0(v,w)}{\|v\|_{\boldsymbol{\mathcal{X}}}\|w\|_{\boldsymbol{\mathcal{Y}}}} \geq \gamma_0 \ . \tag{2.11}$$

2. *The "fluctuation" operators $\{G_j\}_{j\geq 1}$ are small with respect to G_0 in the following sense: there exists a constant $0 < \kappa < 1$ such that*

$$\sum_{j\geq 1} \|G_j\|_{\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}'} \leq \kappa\gamma_0 \ . \tag{2.12}$$

We remark that with (2.10), (2.11), condition (2.12) follows from

$$\sum_{j\geq 1} \|G_0^{-1} G_j\|_{\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{X}}} \leq \kappa \ . \tag{2.13}$$

We show next that, under Assumption 2, the parametric family $G(\sigma)$ is boundedly invertible *uniformly* with respect to the parameter vector σ belonging to the parameter domain $\mathcal{S} = [-1,1]^{\mathbb{N}}$.

Theorem 2. *Under Assumption 2, for every realization $\sigma \in \mathcal{S} = [-1,1]^{\mathbb{N}}$ of the parameter vector, the parametric operator $G(\sigma)$ is boundedly invertible. Specifically, for the bilinear form $\mathcal{G}(\sigma;\cdot,\cdot) : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} \to \mathbb{R}$ associated with $G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}')$, there hold the uniform inf-sup conditions with $\gamma = (1-\kappa)\gamma_0 > 0$ and with κ, $\gamma_0 > 0$ as in Assumption 2,*

$$\forall \sigma \in \mathcal{S}: \quad \inf_{0\neq v\in\boldsymbol{\mathcal{X}}} \sup_{0\neq w\in\boldsymbol{\mathcal{Y}}} \frac{\mathcal{G}(\sigma;v,w)}{\|v\|_{\boldsymbol{\mathcal{X}}}\|w\|_{\boldsymbol{\mathcal{Y}}}} \geq \gamma \tag{2.14}$$

and

$$\forall \sigma \in \mathcal{S}: \qquad \inf_{0\neq w\in\boldsymbol{\mathcal{Y}}} \sup_{0\neq v\in\boldsymbol{\mathcal{X}}} \frac{\mathcal{G}(\sigma;v,w)}{\|v\|_{\boldsymbol{\mathcal{X}}}\|w\|_{\boldsymbol{\mathcal{Y}}}} \geq \gamma \ . \tag{2.15}$$

In particular, for every $g \in \boldsymbol{\mathcal{Y}}'$ and for every $\sigma \in \mathcal{S}$, the parametric operator equation

$$\textit{find } q(\sigma) \in \boldsymbol{\mathcal{X}}: \quad \mathcal{G}(\sigma; q(\sigma), v) = \langle g, v\rangle_{\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Y}}'} \quad \forall v \in \boldsymbol{\mathcal{Y}} \tag{2.16}$$

admits a unique solution $q(\sigma)$ *which satisfies the a-priori estimate*

$$\sup_{\sigma\in\mathcal{S}} \|q(\sigma)\|_{\boldsymbol{\mathcal{X}}} \le \frac{\|g\|_{\boldsymbol{\mathcal{Y}}'}}{(1-\kappa)\gamma_0} \ . \tag{2.17}$$

Proof. As the result is essentially a perturbation result, there are several ways to prove it. One approach, which was used for example in [G], is based on a Neumann Series argument. We give an alternative proof by verifying the inf-sup conditions directly. The inf-sup condition (2.2) is equivalent to the following assertion: given $v \in \boldsymbol{\mathcal{X}}$, there exists $w_v \in \boldsymbol{\mathcal{Y}}$ (linearly depending on v) such that i) $\|w_v\|_{\boldsymbol{\mathcal{Y}}} \le c_1\|v\|_{\boldsymbol{\mathcal{X}}}$ and ii) $\mathcal{G}(v, w_v) \ge c_2\|v\|^2_{\boldsymbol{\mathcal{X}}}$. Then (2.2) holds with $\gamma = c_2/c_1$.

By Assumption 2, in particular by (2.10), i) and ii) are satisfied for the bilinear form $\mathcal{G}_0(\cdot,\cdot)$ with constants $c_{1,0}$ and $c_{2,0}$, i.e., $\gamma_0 = c_{2,0}/c_{1,0}$.

With $v \in \boldsymbol{\mathcal{X}}$ arbitrary and with $w_v \in \boldsymbol{\mathcal{Y}}$ as in i) and ii) for the bilinear form $\mathcal{G}_0(\cdot,\cdot)$ (in particular, independent of σ), we obtain for every $\sigma \in \mathcal{S} = [-1,1]^{\mathbb{N}}$

$$\begin{aligned}
\mathcal{G}(\sigma; v, w_v) &= \mathcal{G}_0(v, w_v) + \sum_{j\ge1} \sigma_j \mathcal{G}_j(v, w_v) \\
&\ge c_{2,0}\|v\|^2_{\boldsymbol{\mathcal{X}}} - \sum_{j\ge1} |\mathcal{G}_j(v, w_v)| \\
&= c_{2,0}\|v\|^2_{\boldsymbol{\mathcal{X}}} - c_{1,0}\sum_{j\ge1} \|G_j\|_{\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}'}\|v\|^2_{\boldsymbol{\mathcal{X}}} \\
&= \left(c_{2,0} - c_{1,0}\sum_{j\ge1} \|G_j\|_{\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}'} \right) \|v\|^2_{\boldsymbol{\mathcal{X}}} \\
&\ge c_{2,0}(1-\kappa)\|v\|^2_{\boldsymbol{\mathcal{X}}} \\
&\ge \tfrac{c_{2,0}}{c_{1,0}}(1-\kappa)\|v\|_{\boldsymbol{\mathcal{X}}}\|w_v\|_{\boldsymbol{\mathcal{Y}}} \\
&= \gamma_0(1-\kappa)\|v\|_{\boldsymbol{\mathcal{X}}}\|w_v\|_{\boldsymbol{\mathcal{Y}}} \ .
\end{aligned}$$

This implies (2.14). The stability condition (2.15) is verified analogously. The a-priori bound (2.17) follows then from (2.5) with the constant $\gamma = (1-\kappa)\gamma_0$. ◻

From the preceding considerations, the following is readily verified.

COROLLARY 3. *Let the affine parametric operator family of operators* $\{G_j\}_{j\ge0}$ *in* (2.9) *satisfy Assumption 2. Then they satisfy Assumption 1 with*

$$C_0 = \frac{1}{(1-\kappa)\gamma_0} \quad \textit{and} \quad b_j := \frac{\|G_j\|_{\boldsymbol{\mathcal{X}}\to\boldsymbol{\mathcal{Y}}'}}{(1-\kappa)\gamma_0} \quad \textit{for all } j \ge 1$$

for any $0 < \mathfrak{p} \le 1$.

2.3. Analytic dependence of solutions. We now establish that the dependence of the solution $q(\sigma)$ on σ is analytic, with precise bounds on the growth of the partial derivatives. This result generalizes the statement for parametric diffusion problems from [CDS1], Theorem 4, to general saddle point problems. It later allows us to prove a-priori estimates for finite approximations of $q(\sigma)$ with respect to σ.

THEOREM 4. *Let the parametric operator family* $\{G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}') : \sigma \in \mathcal{S}\}$ *satisfy Assumption 1 for some* $0 < \mathfrak{p} \le 1$. *Then, for every* $f \in \boldsymbol{\mathcal{Y}}'$ *and every* $\sigma \in \mathcal{S}$ *there exists a unique solution* $q(\sigma) \in \boldsymbol{\mathcal{X}}$ *of the parametric operator equation*

$$G(\sigma)\, q(\sigma) = f \qquad \textit{in } \boldsymbol{\mathcal{Y}}'. \tag{2.18}$$

The parametric solution family $q(\sigma)$ depends analytically on the parameters, and the partial derivatives of the parametric solution family $q(\sigma)$ satisfy the bounds

$$\sup_{\sigma\in\mathcal{S}} \|(\partial_\sigma^\nu q)(\sigma)\|_{\boldsymbol{\mathcal{X}}} \le C_0 \|f\|_{\boldsymbol{\mathcal{Y}}'} |\nu|!\, \tilde{b}^\nu \quad \textit{for all } \nu \in \mathfrak{F}, \tag{2.19}$$

where $0! := 1$ *and the sequence* $\tilde{b} = (\tilde{b}_j)_{j\ge 1} \in \ell^{\mathfrak{p}}(\mathbb{N})$ *is defined by*

$$\tilde{b}_j = b_j / \ln 2 \quad \textit{for all } j \in \mathbb{N}.$$

Proof. Rather than proving (2.19), we prove the (slightly) stronger bound

$$\sup_{\sigma\in\mathcal{S}} \|(\partial_\sigma^\nu q)(\sigma)\|_{\boldsymbol{\mathcal{X}}} \le C_0 \|f\|_{\boldsymbol{\mathcal{Y}}'}\, d_{|\nu|} b^\nu \quad \text{for all } \nu \in \mathfrak{F}\ , \tag{2.20}$$

where the sequence $d = (d_n)_{n\ge 0}$ is defined recursively by

$$d_0 := 1\ , \quad d_n := \sum_{i=0}^{n-1} \binom{n}{i} d_i\ , \quad n = 1, 2, \dots\ . \tag{2.21}$$

The proof of (2.20) proceeds by induction with respect to $|\nu|$: if $|\nu| = 0$, $\nu = 0$ and the assertion (2.20) follows from (2.7) and the a-priori bound (2.5). For $0 \ne \nu \in \mathfrak{F}$, we take the derivative ∂_σ^ν of the equation (2.18). The existence of these derivatives follows as in the proof of Theorem 4.2 in [CDS1] since $G(\sigma)$ is a linear operator from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}'$ and boundedly invertible for every σ. Recalling for the (finitely supported) multi-indices $\nu, \mu \in \mathfrak{F}$ their associated (finite) index sets $\mathfrak{n}, \mathfrak{m} \subset \mathbb{N}$ and abbreviate $n := |\mathfrak{n}| = |\nu|$, $m := |\mathfrak{m}| = |\mu|$, respectively, we find with the generalized product rule due to the σ-independence of f the identity

$$\sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})} \partial_\sigma^{\mathfrak{n}\setminus\mathfrak{m}} (G(\sigma))\, \partial_\sigma^{\mathfrak{m}} (q(\sigma)) = 0 \qquad \text{for all } \sigma \in \mathcal{S}\ .$$

Here, $\mathfrak{P}(\mathfrak{n})$ denotes the power set of $\mathfrak{n} \subset \mathbb{N}$. Solving this identity for $\partial_\sigma^{\mathfrak{n}}(q(\sigma))$, we find

$$G(\sigma)(\partial_\sigma^{\mathfrak{n}} q)(\sigma) = -\sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} \partial_\sigma^{\mathfrak{n}\setminus\mathfrak{m}} (G(\sigma))\, \partial_\sigma^{\mathfrak{m}} (q(\sigma)) \quad \text{in} \quad \boldsymbol{\mathcal{Y}}'\ .$$

From the bounded invertibility of $G(\sigma)$, we get the recursion

$$(\partial_\sigma^\nu q)(\sigma) = -\sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} (G(\sigma))^{-1} \partial_\sigma^{\mathfrak{n}\setminus\mathfrak{m}} (G(\sigma))\, \partial_\sigma^{\mathfrak{m}} (q(\sigma)) \quad \text{in} \quad \boldsymbol{\mathcal{Y}}'\ . \tag{2.22}$$

Taking the $\|\cdot\|_{\boldsymbol{\mathcal{X}}}$ norm on both sides and using the triangle inequality, we find

$$\begin{aligned} \|(\partial_\sigma^\nu q)(\sigma)\|_{\boldsymbol{\mathcal{X}}} &\le \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} \|(G(\sigma))^{-1} \partial_\sigma^{\mathfrak{n}\setminus\mathfrak{m}} (G(\sigma))\|_{\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{X}})} \|\partial_\sigma^{\mathfrak{m}} (q(\sigma))\|_{\boldsymbol{\mathcal{X}}} \\ &\le \sum_{m=0}^{n-1} \sum_{\substack{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n}) \\ |\mathfrak{m}|=m}} \|(G(\sigma))^{-1} \partial_\sigma^{\mathfrak{n}\setminus\mathfrak{m}} (G(\sigma))\|_{\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{X}})} \|\partial_\sigma^{\mathfrak{m}} (q(\sigma))\|_{\boldsymbol{\mathcal{X}}}. \end{aligned} \tag{2.23}$$

Now (2.20) for $n = |\nu| = 1$ follows directly, upon using (2.8) for the singleton sets $\mathfrak{n} = \{j\}$.

We now proceed by induction with respect to $|\nu|$. We consider $\nu \in \mathfrak{F}$ such that $n = |\nu| \geq 2$ and assume that the assertion (2.20) has already been proved for all $\tilde{\nu} \in \mathfrak{F}$ such that $1 \leq |\tilde{\nu}| < n$. We then obtain from (2.23)

$$\begin{aligned} \|(\partial_\sigma^\nu q)(\sigma)\|_{\boldsymbol{\mathcal{X}}} &\leq \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} \|(G(\sigma))^{-1}\partial_\sigma^{\mathfrak{n}\setminus\mathfrak{m}}(G(\sigma))\|_{\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{X}})}\|\partial_\sigma^{\mathfrak{m}}(q(\sigma))\|_{\boldsymbol{\mathcal{X}}} \\ &\leq \sum_{m=0}^{n-1}\sum_{\substack{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\\ |\mathfrak{m}|=m}} C_0\|f\|_{\boldsymbol{\mathcal{Y}}'}\, b^{\nu-\mu} d_m b^\mu \\ &= C_0\|f\|_{\boldsymbol{\mathcal{Y}}'}\, b^\nu \sum_{m=0}^{n-1}\binom{n}{m} d_m \\ &= C_0\|f\|_{\boldsymbol{\mathcal{Y}}'}\, b^\nu d_n \end{aligned}$$

which is (2.20) for $\nu \in \mathfrak{F}$ such that $|\nu| = n$.

The assertion (2.19) now follows from (2.20) and the elementary inequality

$$d_n \leq \left(\frac{1}{\ln 2}\right)^n n! \qquad \text{for all } n \in \mathbb{N}\,. \qquad \square$$

2.4. Convergence rates of N-term gpc (generalized polynomial chaos) approximation. The estimates (2.19) of the partial derivatives of $q(\sigma)$ with respect to σ will be the basis for quantifying approximability of $q(\sigma)$ in the space $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$. This is the space of all functions v for which the norm $\|v\|_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})}$ given by

$$\|v\|^2_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} := \int_{\mathcal{S}} \|v(\sigma)\|^2_{\boldsymbol{\mathcal{X}}}\, d\rho(\sigma) \tag{2.24}$$

is finite. Recall that $\sigma \in \mathcal{S}$ is possibly infinite. The ultimate goal is to *compute* an approximation $\tilde{q}_N$ of q with at most N degrees of freedom such that

$$\|q - \tilde{q}_N\|_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} \lesssim N^{-r} \tag{2.25}$$

with some largest possible rate $r > 0$, where the symbol $\lesssim$ denotes inequality up to constants which are specifically independent of N. If we are able to establish such an estimate with a rate r exceeding the Monte-Carlo benchmark rate $1/2$, this approximation would converge faster than Monte-Carlo methods (assuming equal, uniform cost for the solution of each control problem) even in the case of only finitely many parameters σ. A second key aspect of the methods proposed here is that they allow for reduced resolution in x and t in large parts of the parameter space, in contrast to Monte-Carlo sampling which mandates the same level of resolution in all sampling points.

The first step in the direction to obtain (2.25) will be to establish an *a-priori-type error estimate* for a *best N-term approximation* q_N of q. This means that q_N possesses at most N degrees of freedom and minimizes the error to q with respect to $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$. Of course, this approximation is tied to the choice of the concrete finite-dimensional subspace of $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$ and its basis. In the context of stochastic parameters, the *spectral approach* introduced in [GS], see also [KX, Sch], is based on so-called *Wiener/generalized polynomial chaos (gpc) expansions*. These expansions are performed in terms of tensor products of *orthonormal polynomials* with respect to $L^2(-1,1)$ and the measure (2.6). The choice of this interval stems from choosing the parameter space $\mathcal{S} = [-1,1]^{\mathbb{N}}$; the notion of 'chaos' indicates that this was originally designed to transform stochastic coefficients into deterministic ones, and 'generalized' expresses that one works with orthogonal polynomials with respect to the inner product for $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$ weighted by the uniform probability measure $d\rho(\sigma)$ defined in (2.6).

There are at least three reasons to choose such spectral approaches over others. First, since it is a spectral expansion in terms of orthogonal polynomials, one can achieve an *exponential convergence rate* in the polynomial degree N which is optimal. This result was first established in [BTZ] for the case of a finite number K of random parameters σ for elliptic PDEs with random inputs; here, however, the constant in the error estimate depended strongly on K. Thus, as $K \to \infty$ in the convergence analysis, this approximation result required an assumption of truncation with respect to K. Optimal error estimates for the general case of infinitely many parameters were for the first time established in [ST] for the same problem class under the assumption that the Karhunen-Loève expansion of the stochastic diffusion coefficient decays exponentially to zero in the L^∞ norm. This was relaxed in [CDS1] to the more realistic assumption of algebraic decay, as in (2.19).

The second reason for choosing the spectral approach is that this allows for an immediate computation of the mean field $\mathsf{E}(q_N)$, as detailed below in (2.37). Thirdly, all the inner products in relation to σ for different polynomial degrees vanish; an enormous computational advantage. In our setting, we base our approximations on Legendre polynomials which provide an orthonormal polynomial basis for $L^2(-1,1)$ and which are easy to construct.

To this end, let $L_n(t)$ denote the Legendre polynomial of degree $n \ge 0$ in $(-1,1)$ which is normalized such that

$$\int_{-1}^{1} |L_n(t)|^2 \, \frac{dt}{2} = 1 \; . \tag{2.26}$$

Then $L_0 = 1$ and $\{L_n\}_{n\ge 0}$ is an orthonormal basis of $L^2(-1,1)$ with respect to the measure (2.6). For $\nu \in \mathfrak{F}$, denote $\nu! = \nu_1!\,\nu_2! \ldots$ and introduce the *tensorized Legendre polynomials*

$$L_\nu(\sigma) = \prod_{j\ge 1} L_{\nu_j}(\sigma_j) \; . \tag{2.27}$$

Note that for each $\nu \in \mathfrak{F}$, there are only finitely many nontrivial factors in this product, and each $L_\nu(\sigma)$ depends only on finitely many of the σ_j. By construction, the countable collection $\{L_\nu(\sigma) : \nu \in \mathfrak{F}\}$ is a Riesz basis, i.e., a dense, orthonormal family in $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$: in particular, each $v \in L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$ admits an orthonormal expansion

$$v(\sigma) = \sum_{\nu\in\mathfrak{F}} v_\nu L_\nu(\sigma) \; , \qquad \text{where} \quad v_\nu := \int_{\mathcal{S}} v(\sigma) L_\nu(\sigma) d\rho(\sigma) \in \boldsymbol{\mathcal{X}} \tag{2.28}$$

and there holds Parseval's equality

$$\|v\|^2_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} = \sum_{\mu\in\mathfrak{F}} \|v_\mu\|^2_{\boldsymbol{\mathcal{X}}} \; . \tag{2.29}$$

The Legendre representation (2.28) is the basis for the analysis of best N-term approximation rates. To this end, denote by $\Lambda \subset \mathfrak{F}$ a subset of cardinality $N = \#\Lambda < \infty$, and let $X_\Lambda := \mathrm{span}\{L_\nu : \nu \in \Lambda\} \subset L^2(\mathcal{S}, \rho; \boldsymbol{\mathcal{X}})$. Then, with q_ν denoting the Legendre coefficients of the solution q of the parametric operator equation (2.18), Parseval's identity (2.29) implies

$$\begin{aligned}\Big\|q - \sum_{\nu\in\Lambda} q_\nu L_\nu\Big\|^2_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} &= \inf_{v_\Lambda \in X_\Lambda} \|q - v_\Lambda\|^2_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} \\ &= \sum_{\nu\notin\Lambda} \|q_\nu\|^2_{\boldsymbol{\mathcal{X}}} \,. \end{aligned} \tag{2.30}$$

Best N-term approximation rates in $\|\cdot\|_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})}$ will therefore follow from summability of the norms $\alpha_\nu = \|q_\nu\|_{\boldsymbol{\mathcal{X}}}$ of the Legendre coefficients by the following lemma whose's proof is elementary [CDS2] and which is by now known as *Stechkin's Lemma*.

LEMMA 5. *Let $0 < \mathfrak{p} \le \mathfrak{q} \le \infty$ and $\alpha = (\alpha_\nu)_{\nu\in\mathfrak{F}}$ be a sequence in $\ell^{\mathfrak{p}}(\mathfrak{F})$. If $\mathfrak{F}_N$ is the set of indices corresponding to the N largest values of $|\alpha_\nu|$, we have*

$$\Big(\sum_{\nu\notin\mathfrak{F}_N} |\alpha_\nu|^{\mathfrak{q}}\Big)^{1/\mathfrak{q}} \le \|\alpha\|_{\ell^{\mathfrak{p}}(\mathfrak{F})}\, N^{-r},$$

where $r := \frac{1}{\mathfrak{p}} - \frac{1}{\mathfrak{q}} \ge 0$.

We therefore need to address the $\mathfrak{p}$-summability of the $\|\cdot\|_{\boldsymbol{\mathcal{X}}}$ norms of the Legendre coefficients q_ν of $q(\sigma)$. We first prove estimates for these coefficients.

PROPOSITION 6. *Let $0 < \mathfrak{p} \le 1$ and $b = (b_j)_{j\ge1}$ be as in Assumption 1 above. Moreover, let the sequence $d = (d_j)_{j\ge1}$ be defined by $d_j := \beta b_j$ where $\beta = 1/(\sqrt{3}\ln 2)$, and $\tilde{b} = (\tilde{b}_j)_{j\ge1}$ be defined by $\tilde{b}_j := b_j/\ln 2$. Under Assumption 1, we then have for all $\nu \in \mathfrak{F}$*

$$\|q_\nu\|_{\boldsymbol{\mathcal{X}}} \le C_0 \|f\|_{\boldsymbol{\mathcal{Y}}'} \frac{|\nu|!}{\nu!}\, d^\nu \tag{2.31}$$

and

$$\|q_\nu\|_{\boldsymbol{\mathcal{X}}}\, \|L_\nu\|_{L^\infty(\mathcal{S})} \le C_0 \|f\|_{\boldsymbol{\mathcal{Y}}'} \frac{|\nu|!}{\nu!}\, \tilde{b}^\nu \,. \tag{2.32}$$

Proof. In view of the representation (2.28) in terms of Legendre polynomials, the expansion coefficients q_ν of the solution $q(\sigma)$ of (2.18) read for any $\nu \in \mathfrak{F}$

$$q_\nu = \int_{\mathcal{S}} q(\sigma)\, L_\nu(\sigma)\, d\rho(\sigma) \in \boldsymbol{\mathcal{X}}. \tag{2.33}$$

Since $q(\sigma)$ depends analytically on σ, we can use repeated integration by parts to each of the one-dimensional integrals in (2.33), see the proof of Corollary 6.1 in [CDS1], to arrive at the a-priori estimate

$$\|q_\nu\|_{\boldsymbol{\mathcal{X}}} \le \frac{\beta^{|\nu|}}{\nu!} \sup_{\sigma\in\mathcal{S}} \|(\partial^\nu_\sigma q)(\sigma)\|_{\boldsymbol{\mathcal{X}}} \,.$$

Among others, such estimates allow to steer anisotropic sparse interpolation algorithms of Smolyak type.

Applying (2.19) to further estimate the right hand side immediately yields (2.31). Similarly, also the estimate (2.32) follows. ∎

This result means that the sequence of the norms of the Legendre coefficients $(\|q_\nu\|_{\boldsymbol{\mathcal{X}}})_{\nu\in\mathfrak{F}} \in \ell^{\mathfrak{p}}(\mathfrak{F})$ for the same value of the regularity parameter $\mathfrak{p} \in (0,1]$ for which $G(\sigma)$ satisfies Assumption 1. Since $\mathfrak{p} \le 1$, such sequences are called *sparse* and the corresponding expansions of the type (2.28) *sparse generalized polynomial chaos (gpc) expansions.* The terminology of sparsity has been used widely in the context of compressed sensing, addressing the efficient approximation of random sequences with sequences in $\ell^{\mathfrak{p}}$ with $\mathfrak{p}$ as close as possible to zero, see, e.g., [RW].

Based on the estimates in Proposition 6, we obtain the following result on convergence rates of best N-term polynomial approximations of the parametric solution $q(\sigma)$ of the parametric operator equation (2.18).

Theorem 7. *Under Assumption 1 with some $0 < \mathfrak{p} \le 1$, there exists a sequence $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathfrak{F}$ of index sets whose cardinality does not exceed N such that*

$$\|q - q_N\|_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} \lesssim N^{-r}, \quad r = \frac{1}{\mathfrak{p}} - \frac{1}{2} \tag{2.34}$$

where the inequality holds with a constant independent of N. Here, $q_N := q_{\Lambda_N}$ where q_{Λ_N} denotes the sequence in $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$ whose entries q_ν equal those of the sequence q if $\nu \in \Lambda_N \subset \mathfrak{F}$ and which equal zero otherwise.

In other words, if the parametric operator family $\{G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}') : \sigma \in \mathcal{S}\}$ satisfies Assumption 1 just for weakest case $\mathfrak{p} = 1$, this spectral approach achieves the same benchmark rate as Monte-Carlo methods. For any $\mathfrak{p} < 1$, the rate of the spectral Galerkin approximation will already outperform Monte-Carlo methods. (Since the latter performs approximations for the mean field, the corresponding statement for the expectation values is derived in Corollary 8 below.) In the extreme case that Assumption 1 holds for any $0 \le \mathfrak{p} < 1$ as in the case of affine parameter dependence discussed in the second part of Section 2.2, see Corollary 3, the above rate r is arbitrarily high.

Proof. The proof of Theorem 7 proceeds along the lines of the argument in [CDS1] for the parametric diffusion problem: we use the bounds (2.31) and (2.32), and Theorem 7.2 of [CDS1], i.e.,

$$\text{for } 0 < \mathfrak{p} \le 1: \quad \left(\frac{|\nu|!}{\nu!}\alpha^\nu\right)_{\nu\in\mathfrak{F}} \in \ell^{\mathfrak{p}}(\mathfrak{F}) \quad \text{if and only if} \quad \|\alpha\|_{\ell^1(\mathbb{N})} < 1 \text{ and } \alpha \in \ell^{\mathfrak{p}}(\mathbb{N}).$$

Applying this result to the sequences $\alpha = d$ and to $\alpha = \tilde{b} = (\ln 2)^{-1}b$, we obtain the $\mathfrak{p}$-summability, and, by referring to the Stechkin Lemma 5 with $\mathfrak{q} = 2$ and the Parseval identity (2.29), the assertion (2.34) follows. ∎

At this point, we shall relate the above approximation result to the original variational problem (2.18). Formulated in $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$, the problem reads as follows: find $q \in L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$ such that

$$\mathsf{E}(G(\sigma)q(\sigma)) = f \tag{2.35}$$

where the expectation is defined as

$$\mathsf{E}(G(\sigma)q(\sigma)) := \int_{\mathcal{S}} G(\sigma)q(\sigma)\,d\rho(\sigma). \tag{2.36}$$

With the approximation result of Theorem 7, we can immediately get a corresponding error estimate for the *Galerkin approximation* $q_\Lambda \in X_\Lambda$ of (2.35), i.e., the solution of (2.35) projected onto X_Λ, by applying Céa's Lemma, see, e.g., [BF], in the first step in (2.30). Also, we can derive from q_N an approximation to the mean field or (formal) "ensemble average" $\overline{q} := \mathsf{E}(q)$ as

$$\overline{q_N} := \mathsf{E}(q_N) = \sum_{\nu\in\Lambda_N} e_\nu q_\nu . \tag{2.37}$$

Here q_ν are the expansion coefficients defined in (2.33) and e_ν are the *νth moments* of the Legendre polynomials $e_\nu := \mathsf{E}(L_\nu(\sigma)) = \int_{\mathcal{S}} L_\nu(\sigma)d\rho(\sigma)$. Since we are dealing with the uniform probability measure (2.6), the orthogonality and normalization properties of the Legendre polynomials (2.26) yield $e_\nu = 0$ for all ν except $\nu = 0$ in which case we have $e_0 = 1$. Thus, the expectation of q_N is just given by the 0th polynomial chaos expansion coefficient,

$$\overline{q_N} = \mathsf{E}(q_N) = q_0 = \int_{\mathcal{S}} q(\sigma)d\rho(\sigma). \tag{2.38}$$

Moreover, as in [CDS1], one has by the triangle and the Cauchy-Schwarz inequality the estimate

$$\|\overline{q} - \overline{q_N}\|_{\boldsymbol{\mathcal{X}}} \le \int_{\mathcal{S}} \|q(\sigma,\cdot) - q_N(\sigma,\cdot)\|_{\boldsymbol{\mathcal{X}}} d\rho(\sigma) \le \|q - q_N\|_{L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})} \tag{2.39}$$

which, together with the main approximation estimate (2.34) yields the same rate also for the mean fields.

Corollary 8. *Let the operator family $\{G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}') : \sigma \in \mathcal{S}\}$ satisfy Assumption 1 for some $0 < \mathfrak{p} \le 1$. Then, with the approximation q_N defined in Theorem 7, it holds*

$$\|\overline{q} - \overline{q_N}\|_{\boldsymbol{\mathcal{X}}} \lesssim N^{-r}, \quad r = \frac{1}{\mathfrak{p}} - \frac{1}{2}. \tag{2.40}$$

We next illustrate the scope of the foregoing abstract results with several concrete instances of PDE-constrained control problems: we consider problems constrained by parametric elliptic or parabolic PDE operators and different types of controls. In either case, we develop gpc approximation results by identifying the parametric control problem as particular case of the abstract parametric saddle point problem (2.18). Importantly, due to our formulation as a saddle point problem, the best N-term approximation rates obtained from Theorem 7 pertain to *concurrent* N-term approximation of state and control with *the same set of active gpc coefficients*.

3. Parametric Linear-Quadratic Elliptic Control Problems. We describe the setup of the control problem constrained by a linear parametric elliptic PDE by first addressing conditions on the PDE constraint as an operator equation with a parametric linear elliptic operator $A = A(\sigma)$ on a reflexive Banach space Y. Our standard example with be a scalar diffusion problem. As mentioned above in Section 2.2, this means that we are actually in the case of *affine parameter dependence* so that Assumption 1 always holds for any $0 < \mathfrak{p} \le 1$, as stated in Corollary 3. However, since all the subsequent results also hold for the more general case as long as Assumption 1 as satisfied, we formulate the main results for the more general situation.

Assumption 3. *For each fixed $\sigma \in \mathcal{S}$, the operator $A(\sigma) \in \mathcal{L}(Y, Y')$ is symmetric and boundedly invertible, i.e., $A(\sigma) : Y \to Y'$ is linear, self-adjoint, invertible and* *satisfies the continuity and coercivity estimates*

$$\big|\langle v, A(\sigma)w\rangle_{Y\times Y'}\big| \le C_A\|v\|_Y\|w\|_Y, \qquad v,w\in Y, \tag{3.1}$$
$$\langle v, A(\sigma)v\rangle_{Y\times Y'} \ge c_A\|v\|_Y^2, \qquad v\in Y, \tag{3.2}$$

with some constants $0 < c_A \le C_A < \infty$ *independent of* σ*.*

These imply the estimates

$$c_A\|w\|_Y \le \|A(\sigma)w\|_{Y'} \le C_A\|w\|_Y \quad \text{for any } w\in Y \tag{3.3}$$

which, in terms of operator norms, may be expressed as

$$\|A(\sigma)\|_{Y\to Y'} := \sup_{w\in Y,\ w\not\equiv 0}\frac{\|A(\sigma)w\|_{Y'}}{\|w\|_Y} \le C_A, \qquad \|A(\sigma)^{-1}\|_{Y'\to Y} \le c_A^{-1}. \tag{3.4}$$

If the precise format of the constants in (3.3) does not matter, we will abbreviate this as

$$\|A(\sigma)w\|_{Y'} \sim \|w\|_Y \quad \text{for any } w\in Y \tag{3.5}$$

and use $a \lesssim b$ or $a \gtrsim b$ for the corresponding one-sided estimates.

Some examples of operators A and space Y are provided next which satisfy Assumption 3 provided that (3.6) stated below holds. In all the following, $\Omega \subset \mathbb{R}^{\mathfrak{d}}$ denotes a bounded domain with Lipschitz boundary $\partial\Omega$.

Example 9.

(i) *(Dirichlet problem with homogeneous Dirichlet boundary conditions)* $\langle v, A(\sigma)w\rangle_{Y\times Y'} = \int_\Omega (a(\sigma)\nabla_x v\cdot\nabla_x w)dx$, $Y = H_0^1(\Omega)$. *In this and all the following examples, the coefficient* $a(\sigma)$ *is supposed to satisfy the* uniform ellipticity assumption UEA(r_a, R_a)*: there exist positive constants* r_a, R_a *such that for all* $x\in\Omega$ *and all* $\sigma\in\mathcal{S}$ *it holds*

$$0 < r_a \le a(x,\sigma) \le R_a < \infty. \tag{3.6}$$

This is a standard assumption for stochastic PDEs since it guarantees that the operator $A(\sigma)$ *is elliptic uniformly with respect to the parameter sequences* σ*. More generally,* r_a *and* R_a *may depend on* σ*, but are in* $L^p(\mathcal{S};\rho)$ *for* $p\ge 2$*, as is the case for a log-normal distribution, see, e.g., [BNT] for examples.*

(ii) *(Reaction-diffusion problem with possibly anisotropic diffusion with Neumann boundary conditions)*

$$\langle v, A(\sigma)w\rangle_{Y\times Y'} = \int_\Omega (a(\sigma)\nabla_x v\cdot\nabla_x w + vw)dx, \quad Y = H^1(\Omega)\ .$$

Note that Assumption 3 is, due to the self-adjointness a special case of the conditions on the operator G in Proposition 1 with $\mathcal{X} = \mathcal{Y} = Y$. Thus, this assumption implies that for any given deterministic $f\in Y'$ and fixed $\sigma\in\mathcal{S}$, the operator equation

$$A(\sigma)\,y = f \tag{3.7}$$

has a unique solution $y = y(\sigma)\in Y$.

3.1. Distributed or Neumann boundary control. Allowing an additional function $u = u(\sigma)$ on the right hand side of (3.7), we ask to steer the solution of such an equation towards a prescribed desired deterministic state y_*, under the condition that the effort on u should be minimal.

We formulate the control problems involving expectation values as, e.g., in [GLL]. As we have seen at the end of Section 2.4, the respective approximations can then be derived from the main approximation result in Theorem 7 as in Corollary 8.

We define an optimal control problem with a functional of *tracking type* as follows: minimize over the *state* $y(\sigma)$ and the *control* $u(\sigma)$ the functional

$$\mathsf{E}(\tilde{J}(y(\sigma), u(\sigma))) := \frac{1}{2}\mathsf{E}(\|\mathfrak{T}y(\sigma) - y_*\|_O^2) + \frac{\omega}{2}\mathsf{E}(\|u(\sigma)\|_U^2) \tag{3.8}$$

subject to the linear operator equation

$$\mathsf{E}(A(\sigma)\, y(\sigma)) = f + \mathfrak{E}\mathsf{E}(u(\sigma)). \tag{3.9}$$

Here $\omega > 0$ is a fixed constant which balances the least squares approximation of the states and the norm for the control and $\mathfrak{T}$, $\mathfrak{E}$ are some linear (trace and extension) operators described below.

We need to add some requirements on the norms used in (3.8). In view of Assumption 3, in order for (3.9) to have a well-defined unique solution, we need to assure that either $y \in Y$ or $\mathfrak{E}u \in Y'$. The latter is satisfied if the *control space* U defining the penalty norm part of the functional is such that $U \subseteq Y'$ with continuous embedding. Then the *observation space* O defining the least squares part of the functional (3.19) may be chosen as any $O \supseteq Y$. In this case, $\mathfrak{T}$ may be any continuous linear operator from Y onto its range, i.e., $\|\mathfrak{T}v\|_{\mathrm{range}(\mathfrak{T})} \lesssim \|v\|_Y$ for $v \in Y$ with range($\mathfrak{T}$) continuously embedded in O. Alternatively, assuring $\mathfrak{T}y \in O$ and selecting $O \subseteq Y$ embedded continuously would allow for any choice of U.

There are two standard examples covered by this formulation which we have in mind (see [DK] for more general formulations). A *distributed control* problem is one where the control is exerted on all of the right hand side of (3.9), i.e., $\mathfrak{E}$ is just the identity. This case is perhaps rather of academic nature but serves as a good illustration for the essential mechanisms.

EXAMPLE 10. *(Dirichlet problem with distributed control)*
Here the PDE constraints (before taking the expectation values on both sides) are given by the standard scalar second order Dirichlet problem with distributed control,

$$\begin{array}{rcll} -\nabla_x\cdot(a(\sigma)\nabla_x)\, y(\sigma) & = & f + u(\sigma) & \textit{in } \Omega, \\ y(\sigma) & = & 0 & \textit{on } \partial\Omega, \end{array} \tag{3.10}$$

which gives rise to the operator equation (3.9) *with*

$$\langle v, A(\sigma)w\rangle_{Y\times Y'} = \int_\Omega a(\sigma)\nabla_x v \cdot \nabla_x w\, dx, \quad Y = H_0^1(\Omega),\ Y' = H^{-1}(\Omega), \tag{3.11}$$

and given $f \in Y'$. Admissible choices for O, U are the classical case $O = U = L_2(\Omega)$, see [L], or the natural choice $O = Y$ and $U = Y'$, in which case the operators $\mathfrak{T}, \mathfrak{E}$ are the canonical injections $\mathfrak{T} = I$, $\mathfrak{E} = I$. Many more possible choices covering, in particular, fractional Sobolev spaces, have been discussed in [DK], as well as including a class of Neumann problems with distributed control.

Example 11. *(Reaction-diffusion problem with Neumann boundary control) Consider the second order Neumann problem in strong form*

$$
\begin{aligned}
-\nabla_x \cdot (a(\sigma)\nabla_x)\, y(\sigma) + y(\sigma) &= f && \text{in } \Omega, \\
(a(\sigma)\nabla_x y(\sigma)) \cdot \mathbf{n} &= u(\sigma) && \text{on } \partial\Omega,
\end{aligned}
\tag{3.12}
$$

where $\mathbf{n}$ *denotes the outward normal at* $\partial\Omega$. *Here the weak form is based on setting* $Y = H^1(\Omega)$ *and*

$$
\langle v, A(\sigma)w\rangle_{Y\times Y'} = \int_\Omega (a(\sigma)\nabla_x v \cdot \nabla_x w + vw)dx,
\tag{3.13}
$$

and given $f \in Y'$. *Recall that for any* $v \in H^1(\Omega)$, *its normal trace* $\mathbf{n}\cdot\nabla_x v$ *to* $\partial\Omega$ *belongs to* $H^{-1/2}(\partial\Omega)$. *Thus, in order for the right hand side of* (3.12) *to be well-defined, the control* u *must belong to* $H^{-1/2}(\partial\Omega)$, *i.e., the operator* $\mathfrak{E}$ *is the adjoint of the normal trace operator, or,* $\mathfrak{E} : H^{-1/2}(\partial\Omega) \to Y'$ *is an extension operator to* Ω. *The formulation of the constraint as an operator equation reads in this case*

$$
A(\sigma)\, y(\sigma) = f + \mathfrak{E}u(\sigma).
\tag{3.14}
$$

As previously, one could choose O *to be a space defined on* Ω. *However, a more frequent practical situation arises when one wants to achieve a prescribed state on some part of the boundary. Denote by* $\Gamma_\circ \subseteq \partial\Omega$ *an* observation boundary *with strictly positive* $\mathfrak{d}-1$*-dimensional measure and by* $\mathfrak{T} : H^1(\Omega) \to H^{1/2}(\Gamma_\circ)$ *the trace operator to this part of the boundary. Then an admissible choice is* $O = H^{1/2}(\Gamma_\circ)$. *As discussed above, we need to require for the control that* $u \in H^{-1/2}(\partial\Omega)$. *For these choices, the functional* (3.8) *is of the form*

$$
\tilde{J}(y,u) = \frac{1}{2}\|\mathfrak{T}y - y_*\|^2_{H^{1/2}(\Gamma_\circ)} + \frac{\omega}{2}\|u\|^2_{H^{-1/2}(\partial\Omega)}.
\tag{3.15}
$$

The fractional trace norms appearing here in a natural form are often replaced, perhaps partly due to the difficulty of evaluating fractional order Sobolev norms numerically, by the classical choice $\Gamma_\circ = \partial\Omega$ *and* $O = U = L_2(\partial\Omega)$ *[L]; we hasten to add, however, that in the context of* multiresolution discretizations in Ω and on $\partial\Omega$, *fractional Sobolev norms can be realized numerically in optimal complexity, see, e.g.,[DK, GK] and [Bu1, Bu2] for corresponding numerical experiments including distributed control problems for elliptic PDEs. Wavelet constructions on general polygonal domains were provided, e.g., in [HaSch, Ng].*

One calls (3.8) with constraints (3.9) a *linear-quadratic control problem*: a quadratic functional is to be minimized subject to a linear equation coupling state and control. From an optimization point of view, the solution of this problem has a simple structure: on account of $\tilde{J}$ being convex, one only needs to consider the first order conditions for optimality. To derive these, for $\sigma \in \mathcal{S}$, in principle, the *dual operator* of $A(\sigma)$ comes into play which is defined by

$$
\langle A(\sigma)^* v, w\rangle_{Y'\times Y} := \langle v, A(\sigma)w\rangle_{Y\times Y'}
\tag{3.16}
$$

that is, $A(\sigma)^* \in \mathcal{L}(Y, Y')$. Of course, since in Assumption 3 $A(\sigma)$ was required to be self-adjoint for each fixed $\sigma \in \mathcal{S}$, we have $A(\sigma)^* = A(\sigma)$.

Note that in case of an unsymmetric $A(\sigma)$, the property to be boundedly invertible (3.5) immediately carries over to $A(\sigma)^*$, that is, for fixed $\sigma \in \mathcal{S}$ and any $v \in Y$, one has the mapping property

$$
\|A(\sigma)^* v\|_{Y'} \sim \|v\|_Y.
\tag{3.17}
$$

For ease of presentation in this paper, we select here the *natural case* $O = Y$ and $U = Y'$ resulting in $\mathfrak{T} = I$ and $\mathfrak{E} = I$ for the trace and extension operators. The more general case which may involve Sobolev spaces with possibly fractional smoothness indices has been treated in [DK] for PDE-constrained control problems without parameters.

To represent the Hilbert space norms in the optimization functional, we shall employ *Riesz* operators $R_Y : Y \to Y'$ defined by

$$\langle v, R_Y w\rangle_{Y\times Y'} := (v,w)_Y, \qquad v,w \in Y. \tag{3.18}$$

Defining $R_{Y'} : Y' \to Y$ correspondingly by $\langle v, R_{Y'} w\rangle_{Y'\times Y} := (v,w)_{Y'}$ for $v,w \in Y'$, this implies $R_{Y'} = R_Y^{-1}$ so that we can write both norms in the target functional in terms of one Riesz operator $R = R_Y$. Since the inner product $(\cdot,\cdot)_Y$ is symmetric, the Riesz operator R is also symmetric. In view of Corollary 8, for easier readability and in view of Corollary 8, we formulate the following result for a fixed choice of σ, i.e., without taking the expectation values as in (3.8) and (3.9), see also Remark 19 below.

Proposition 12. *Necessary and sufficient for the linear-quadratic control problem to minimize for (fixed)* $\sigma \in \mathcal{S}$

$$J(y(\sigma),u(\sigma)) := \frac{1}{2}\|y(\sigma)-y_*\|_Y^2 + \frac{\omega}{2}\|u(\sigma)\|_{Y'}^2 \tag{3.19}$$

over all $(y(\sigma),u(\sigma)) \in Y\times Y'$ *subject to* (3.9) *are the* Euler equations *for the solution triple* $(y(\sigma),p(\sigma),u(\sigma)) \in Y\times Y\times Y'$

$$\text{(EE)} \qquad \begin{aligned} A(\sigma)\,y(\sigma) &= f + u(\sigma) \\ A(\sigma)^*\,p(\sigma) &= R(y_* - y(\sigma)) && (3.20) \\ \omega R^{-1}\,u(\sigma) &= p(\sigma). && (3.21) \end{aligned}$$

Proof. We present a proof of this well-known result only to bring out the roles of the Riesz operators; we skip in this proof the dependence of σ for better readability. The derivation of (EE) is based on computing the zeroes of the first order variations of the Lagrangian functional

$$\mathrm{Lagr}(y,p,u) := J(y,u) + \langle p, A\,y - f - u\rangle_{Y\times Y'}, \tag{3.22}$$

introducing a new variable $p \in Y$ called the *Lagrangian* or *adjoint variable* by which the constraints (3.9) are appended to the functional J, see, e.g., [L]. By inserting definition (3.19) and (3.18), the Lagrangian functional attains the form

$$\mathrm{Lagr}(y,p,u) = \tfrac{1}{2}\,\langle y-y_*, R(y-y_*)\rangle_{Y\times Y'} + \tfrac{\omega}{2}\,\langle u, R^{-1}u\rangle_{Y'\times Y} + \langle p, A\,y - f - u\rangle_{Y\times Y'}\,. \tag{3.23}$$

The constraint (3.9) is recovered as the zero of the first variation of $\mathrm{Lagr}(y,p,u)$ in direction of p. Moreover, $\frac{\partial}{\partial u}\mathrm{Lagr}(y,p,u) = 0$ yields $\omega R^{-1}u - p = 0$. Finally,

$$\begin{aligned} \frac{\partial}{\partial y}\mathrm{Lagr}(y,p,u) &:= \lim_{\delta\to 0}\frac{\mathrm{Lagr}(y+\delta,p,u) - \mathrm{Lagr}(y,p,u)}{\delta} \\ &= \lim_{\delta\to 0}\frac{\frac{1}{2}\,\langle \delta, R(y-y_*)\rangle_{Y\times Y'} + \frac{1}{2}\,\langle y-y_*, R\delta\rangle_{Y\times Y'} + \langle p, A\delta\rangle_{Y\times Y'}}{\delta} \\ &= \lim_{\delta\to 0}\frac{\langle \delta, R(y-y_*)\rangle_{Y\times Y'} + \langle p, A\delta\rangle_{Y\times Y'}}{\delta} \end{aligned}$$

by symmetry of R. Bringing A on the other side of the dual form yields

$$\frac{\partial}{\partial y}\,\mathrm{Lagr}(y,p,u) = R(y-y_*) + A^* p$$

and therefore $\frac{\partial}{\partial y}\,\mathrm{Lagr}(y,p,u) = 0$ if and only if (3.20) holds. ◻

In our formulation, the design equation (3.21) expresses the control just as a weighted Riesz transformed adjoint state. For later analysis, it will help us to eliminate the control using (3.21) and write (EE) as the *condensed Euler equations* for the solution pair $(y(\sigma), p(\sigma)) \in Y \times Y$

$$\begin{aligned} A(\sigma)\,y(\sigma) &= f + \tfrac{1}{\omega} R\,p(\sigma) \\ A(\sigma)^*\,p(\sigma) &= R(y_* - y(\sigma)). \end{aligned} \tag{3.24}$$

With the abbreviation $\hat{y}_* := R\,y_* \in Y'$, we write this as a coupled system to find for given $(f, \hat{y}_*) \in Y' \times Y'$ a solution pair $(y(\sigma), p(\sigma)) \in Y \times Y$ which solves

$$\begin{pmatrix} A(\sigma) & -\frac{1}{\omega} R \\ R & A(\sigma)^* \end{pmatrix} \begin{pmatrix} y(\sigma) \\ p(\sigma) \end{pmatrix} = \begin{pmatrix} f \\ \hat{y}_* \end{pmatrix}. \tag{3.25}$$

Identifying the matrix operator appearing in this system with $G(\sigma)$ in the abstract problem in Section 2, we define the corresponding bilinear form $\mathcal{G}(\sigma;\cdot,\cdot) : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}} \to \mathbb{R}$ where $\boldsymbol{\mathcal{X}} := Y \times Y$, $\boldsymbol{\mathcal{Y}} := \boldsymbol{\mathcal{X}}$, for $q = (y,p)$, $\tilde{q} = (\tilde{y}, \tilde{p}) \in \boldsymbol{\mathcal{X}}$ by

$$\begin{aligned} \mathcal{G}(\sigma; q, \tilde{q}) &:= \left\langle q, \begin{pmatrix} A(\sigma) & -\frac{1}{\omega} R \\ R & A(\sigma)^* \end{pmatrix} \tilde{q} \right\rangle_{\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}}'} \\ &= \langle y, A(\sigma)\tilde{y}\rangle_{Y\times Y'} - \tfrac{1}{\omega}\,\langle y, R\tilde{p}\rangle_{Y\times Y'} + \langle p, R\tilde{y}\rangle_{Y\times Y'} + \langle p, A(\sigma)^*\tilde{p}\rangle_{Y\times Y'}. \end{aligned} \tag{3.26}$$

We equip the space $\boldsymbol{\mathcal{X}}$ with the norm

$$\|q\|_{\boldsymbol{\mathcal{X}}} = \left\| \begin{pmatrix} y \\ p \end{pmatrix} \right\|_{Y\times Y} = \|y\|_Y + \|p\|_Y. \tag{3.27}$$

Proposition 13. *The parametric bilinear form $\mathcal{G}(\sigma;\cdot,\cdot) : \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}} \to \mathbb{R}$ is continuous on $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}}$ for any constant weight $\omega > 0$, and uniformly with respect to the parameter vector σ. It is coercive on $\boldsymbol{\mathcal{X}}$ for $\omega = 1$ with coercivity constant $c_{\mathcal{G}} := \frac{c_A}{2}$ and c_A from* (3.2). *For $0 < \omega \le 1$, it is coercive on $\boldsymbol{\mathcal{X}}$ with a constant $c_{\mathcal{G}}$ defined below in* (3.29). *Moreover, it is symmetric for $\omega = 1$.*

Proof. The symmetry of $\mathcal{G}(\sigma;\cdot,\cdot)$ for $\omega = 1$ follows immediately from the representation (3.26) and by recalling that $A(\sigma)$ is self-adjoint. The continuity of $\mathcal{G}(\sigma;\cdot,\cdot)$ results from the definition of R (3.18) and applying Cauchy-Schwarz inequality, together with the continuity (3.1) of $A(\sigma)$, i.e., for any $q = (y,p)$, $\tilde{q} = (\tilde{y},\tilde{p}) \in \boldsymbol{\mathcal{X}}$ we have from (3.26)

$$\begin{aligned} |\mathcal{G}(\sigma;q,\tilde{q})| &\le \left|\langle y, A(\sigma)\tilde{y}\rangle_{Y\times Y'}\right| + \left|\langle p, A(\sigma)^*\tilde{p}\rangle_{Y\times Y'}\right| + \tfrac{1}{\omega}\,|(y,\tilde{p})_Y| + |(p,\tilde{y})_Y| \\ &\le C_A\left(\|y\|_Y\|\tilde{y}\|_Y + \|p\|_Y\|\tilde{p}\|_Y\right) + \tfrac{1}{\omega}\|y\|_Y\|\tilde{p}\|_Y + \|p\|_Y\|\tilde{y}\|_Y \\ &\le C_A\left(1 + \tfrac{1}{\omega}\right)\|q\|_{\boldsymbol{\mathcal{X}}}\|\tilde{q}\|_{\boldsymbol{\mathcal{X}}} \quad =: C_{\mathcal{G}}\|q\|_{\boldsymbol{\mathcal{X}}}\|\tilde{q}\|_{\boldsymbol{\mathcal{X}}}. \end{aligned} \tag{3.28}$$

As for the coercivity, for $q = (y,p) \in \boldsymbol{\mathcal{X}}$, using the symmetry of R, its definition, the coercivity (3.2) and Cauchy-Schwarz' inequality, we infer for $0 < \omega \le 1$ (meaning that

$(1 - \frac{1}{\omega} \leq 0)$

$$\begin{aligned}\mathcal{G}(\sigma; q, q) &= \langle y, A(\sigma)y\rangle_{Y\times Y'} + \langle p, A(\sigma)^* p\rangle_{Y\times Y'} + \left(1 - \tfrac{1}{\omega}\right)\langle y, Rp\rangle_{Y\times Y'}\\ &\geq c_A\left(\|y\|_Y^2 + \|p\|_Y^2\right) - \left(\tfrac{1}{\omega} - 1\right)|(y,p)_Y|\\ &\geq c_A\left(\|y\|_Y^2 + \|p\|_Y^2\right) - \left(\tfrac{1}{\omega} - 1\right)\|y\|_Y\|p\|_Y .\end{aligned}$$

In case $\omega = 1$, this immediately yields $\mathcal{G}(\sigma; q, q) \geq \frac{c_A}{2}\|q\|^2_{\boldsymbol{\mathcal{X}}} = c_{\mathcal{G}}\|q\|^2_{\boldsymbol{\mathcal{X}}}$ for every $\sigma \in \mathcal{S}$. For $\omega < 1$, we obtain

$$\begin{aligned}\mathcal{G}(\sigma; q, q) &\geq \left(c_A - (\tfrac{1}{\omega} - 1)\right)\left(\|y\|_Y^2 + \|p\|_Y^2\right)\\ &\geq \tfrac{1}{2}\left(c_A - (\tfrac{1}{\omega} - 1)\right)\|q\|^2_{\boldsymbol{\mathcal{X}}} =: c_{\mathcal{G}}\|q\|^2_{\boldsymbol{\mathcal{X}}} .\end{aligned} \tag{3.29}$$

□

By the Theorem of Lax-Milgram, we therefore have, based on Proposition 13, the following result.

THEOREM 14. *Under Assumption 3, for every $0 < \omega \leq 1$ and for every $\sigma \in \mathcal{S}$, the control problem (3.25) admits a unique solution $q(\sigma) = (y(\sigma), u(\sigma)) \in \boldsymbol{\mathcal{X}}$ for any given deterministic right hand side $g := (f, \hat{y}_*) \in \boldsymbol{\mathcal{X}}'$.*

Since, moreover, the parametric family $\{A(\sigma) : \sigma \in \mathcal{S}\}$ depends on σ in a affine fashion, i.e.,

$$A(\sigma) = A_0 + \sum_{j\geq 1}\sigma_j A_j, \tag{3.30}$$

the parametric matrix operator $G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}}')$ satisfies Assumption 2 with $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}} = Y \times Y$.

COROLLARY 15. *State, the costate and the control are* simultaneously analytic *with respect to all parameters. Moreover, the tensorized Legendre expansion (see (2.27)) of the solution triple $(y(\sigma), p(\sigma), u(\sigma))$ is sparse, and therefore, in particular, best N-term gpc approximation rates of all these quantities in the $L^2(\mathcal{S}, \rho; \boldsymbol{\mathcal{X}})$ norm hold.*

Proof. The first part of the statement follows by Theorem 4; the preceding result establishes the *simultaneous analyticity* of state as well as of the costate, with respect to all parameters and therefore, by (3.21), also of the control. The second statement follows upon referring to Theorem 7. □

REMARK 16. *Note that the* affine dependence *of the operator $G(\sigma)$ in (3.26), see Corollary 3, was crucial in being able to use the abstract results of Section 2. Analogous analytic dependence results also hold for control problems with certain more general parameter dependences.*

Occasionally, it is useful to derive from (3.25) an equation for the control alone.

PROPOSITION 17. *Under Assumption 3, system (EE) reduces to the* condensed equation *for the control*

$$\left(A(\sigma)^{-*}RA(\sigma)^{-1} + \omega R^{-1}\right)u(\sigma) = A(\sigma)^{-*}R\left(y_* - A(\sigma)^{-1}f\right) \tag{3.31}$$

(using $A^{-*} := (A^*)^{-1}$) which we abbreviate as

$$M(\sigma)\,u(\sigma) = m(\sigma). \tag{3.32}$$

Proof. On account of Assumption 3, $A(\sigma) \in \mathcal{L}(Y, Y')$ is invertible uniformly with respect to $\sigma \in \mathcal{S}$ so that (3.9) can be expressed as

$$y(\sigma) = A(\sigma)^{-1} f + A(\sigma)^{-1} u(\sigma). \tag{3.33}$$

Inserted into (3.20) this yields

$$A(\sigma)^* p(\sigma) = -RA(\sigma)^{-1} u(\sigma) + R(y_* - A(\sigma)^{-1} f) \tag{3.34}$$

and, again by Assumption 3,

$$p(\sigma) = -A(\sigma)^{-*} RA(\sigma)^{-1} u(\sigma) + A(\sigma)^{-*} R(y_* - A(\sigma)^{-1} f).$$

Using the identity (3.21), we can eliminate $p(\sigma)$ and arrive at

$$\omega R^{-1} u(\sigma) = -A(\sigma)^{-*} RA(\sigma)^{-1} u(\sigma) + A(\sigma)^{-*} R(y_* - A(\sigma)^{-1} f)$$

which is just (3.31). □

REMARK 18. *We observe that the condensed equation* (3.31) *contains the boundedly invertible, parametric Schur complement operator $M(\sigma)$; this operator, while being boundedly invertible, is* not *affine in the parameter vector σ anymore even if $A(\sigma)$ was. Therefore, the theory developed in the second part in Section 2 in the special case of affine parameter dependence does not apply. Nevertheless, analytic parameter dependence can be inferred for $M(\sigma)$ from the structure of its definition, and analytic continuation as in [CDS2] will allow to infer directly analytic dependence and best N-term gpc approximation rates for the control $u(\sigma)$* without *approximation of the state. As this requires introduction of complex extensions of all operators, forms and spaces involved, we do not address this in detail here.*

REMARK 19. *The setup is similar to the control problems considered in [GLL, HLM] where, however, the number of stochastic parameters is* assumed *to be finite.*

We can apply the same techniques used in [GLL] to derive the optimality conditions (EE) for the control problem and PDE constraints with expectation values as in (3.8) *subject to* (3.9) *and arrive at the* expectations of the Euler equations *for the solution triple (y, p, u) as*

$$\text{(EEE)} \qquad \begin{aligned} \mathsf{E}(A(\sigma)\, y(\sigma)) &= f + \mathsf{E}(u(\sigma)) && (3.35)\\ \mathsf{E}(A(\sigma)^*\, p(\sigma)) &= R(y_* - \mathsf{E}(y(\sigma))) && (3.36)\\ \omega R^{-1}\, \mathsf{E}(u(\sigma)) &= \mathsf{E}(p(\sigma)). && (3.37) \end{aligned}$$

However, as explained in the derivation of Corollary 8, we can remain in the situation of considering (EE) and all the following necessary conditions for optimality just as in Proposition 12 for a fixed parameter σ.

3.2. Dirichlet boundary control. The last and perhaps practically most relevant example of control problems with a tracking-type functional and a stationary PDE as constraint concerns problems with Dirichlet boundary control: minimize for some given data y_* the quadratic functional

$$J(y, u) = \frac{1}{2}\|y - y_*\|_O^2 + \frac{\omega}{2}\|u\|_U^2, \tag{3.38}$$

where state y and control u are coupled through the linear elliptic boundary value problem

$$\begin{aligned} -\nabla_x \cdot (a(\sigma)\nabla_x y) + y &= f && \text{in } \Omega,\\ y &= u && \text{on } \Gamma,\\ (a(\sigma)\nabla_x y) \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \setminus \Gamma. \end{aligned} \tag{3.39}$$

Here $\Gamma \subset \partial\Omega$ denotes the *control boundary* assumed to be a set of positive Lebesgue measure on which the control is exerted. Of course, we could allow again for an observation boundary and trace to this boundary in (3.38) as in Example 11, see [K]. We dispense with this generalization here and choose for the following simply $O = H^1(\Omega)$ and given observation $y_* \in H^1(\Omega)$. Recall that our approximation result depends fundamentally on the fact that the operator $G(\sigma)$ is boundedly invertible. Thus, some of the situations causing problems with deterministic L^2 Dirichlet control problems and their finite element discretizations, see, e.g., [CR, GY, KV], do not appear here. Envisaging the discretization of the state/costate/control with respect to space in terms of wavelet coordinates at a later stage, the choice of fractional norms does not pose any numerical difficulty since it just amounts to a multiplication with a diagonal matrix; see [Bu2, Pa] for corresponding numerical studies. Specifically, in [Pa] numerous cases of Dirichlet boundary control problems are studied by varying fractional trace norms for O, U and the parameter ω.

To formulate (3.39) in weak form, we define $A(\sigma)$ as in (3.13), and set $Y = H^1(\Omega)$. It is because of the appearance of the control u as a Dirichlet boundary condition in (3.39) that this is referred to as a *Dirichlet boundary control* problem. As it will be required to allow for repeated updates of the control, this suggests to formulate the constraints (3.39) weakly as a *saddle point problem* itself which results from appending the Dirichlet boundary conditions by Lagrange multipliers as follows. The trace operator to Γ, $\mathfrak{T} : H^1(\Omega) \to H^{1/2}(\Gamma)$ is surjective and defines a bilinear form

$$\langle \mathfrak{T}v, w\rangle_{H^{1/2}(\Gamma)\times(H^{1/2}(\Gamma))'} = \langle \mathfrak{T}v, w\rangle_{H^{1/2}(\Gamma)\times(H^{1/2}(\Gamma))'} \tag{3.40}$$

on $H^1(\Omega) \times (H^{1/2}(\Gamma))'$. Setting $Q := (H^{1/2}(\Gamma))'$, the PDE constraint (3.39) can be formulated weakly as a linear saddle point problem: find $(y_1, y_2) \in Y \times Q$ such that

$$\begin{pmatrix} A(\sigma) & \mathfrak{T}^* \\ \mathfrak{T} & 0 \end{pmatrix} \begin{pmatrix} y_1(\sigma) \\ y_2(\sigma) \end{pmatrix} = \begin{pmatrix} f \\ u(\sigma) \end{pmatrix} \tag{3.41}$$

holds. The trace operator $\mathfrak{T} : Y \to Q$ is continuous and surjective on the kernel of $A(\sigma)$ yielding that the linear saddle point operator

$$B(\sigma) := \begin{pmatrix} A(\sigma) & \mathfrak{T}^* \\ \mathfrak{T} & 0 \end{pmatrix} : Y \times Q \to Y' \times Q' \tag{3.42}$$

is an isomorphism and one has the norm equivalence

$$\left\| B(\sigma) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{Y' \times Q'} \sim \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{Y \times Q}, \tag{3.43}$$

see, e.g., [K]. Thus, if again $A(\sigma)$ satisfies Assumption 3, we have assured that the saddle point operator $B(\sigma)$ for the PDE constraint defined in (3.41) also satisfies Assumption 3. Finally, we choose for the control in (3.38) the natural space $U = H^{1/2}(\Gamma)$. For the control problem to minimize (3.38) subject to (3.41), the optimality conditions, derived analogously as in Proposition 12 are now to find for given $f \in Y'$, $y_* \in Y$ the quintuple $(y_1(\sigma), y_2(\sigma), p_1(\sigma), p_2(\sigma), u(\sigma)) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}} \times Q$ for $\boldsymbol{\mathcal{X}} := Y \times Q$

such that

$$
\text{(DEE)} \qquad \begin{aligned}
B(\sigma)\begin{pmatrix} y_1(\sigma) \\ y_2(\sigma) \end{pmatrix} &= \begin{pmatrix} f \\ u(\sigma) \end{pmatrix} \\
B(\sigma)^*\begin{pmatrix} p_1(\sigma) \\ p_2(\sigma) \end{pmatrix} &= \begin{pmatrix} -R_Y(y(\sigma)-y_*) \\ 0 \end{pmatrix} \\
\omega R_U u(\sigma) &= p_2(\sigma)
\end{aligned}
$$

where R_Y is defined as in (3.18) and R_U accordingly for $(\cdot,\cdot)_U$. Setting $\hat{y}_* := R_Y y_* \in Y'$ and using the design equation in (DEE) to eliminate $p_2(\sigma)$, we arrive at the saddle point system of saddle point problems similar to (3.25), to solve for $y(\sigma), p(\sigma)) := (y_1(\sigma), y_2(\sigma), p_1(\sigma), p_2(\sigma)) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}}$ the system

$$
G(\sigma) := \begin{pmatrix} B(\sigma) & \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{\omega}R_U^{-1} \end{pmatrix} \\ \begin{pmatrix} R_Y & 0 \\ 0 & 0 \end{pmatrix} & B(\sigma)^* \end{pmatrix} \begin{pmatrix} y(\sigma) \\ p(\sigma) \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ \hat{y}_* \\ 0 \end{pmatrix} =: g\,. \tag{3.44}
$$

Corollary 20. *Together with Theorem 14, we have therefore established again the* simultaneous analyticity *of all the solution functions $y(\sigma), p(\sigma), u(\sigma)$ for the case that $A(\sigma)$ depends affinely on σ according to* (3.30). *Moreover, applying again Theorem 7, we have established best N-term gpc approximation rates for the state, costate and control in the $L_2(\mathcal{S}, \rho; \boldsymbol{\mathcal{X}})$ norm with the same rate r.*

4. Parametric Linear-Quadratic Parabolic Control Problems. The preceding control problems were stationary, i.e., the equation of state was *elliptic*. We now show how control problems with *parabolic* equations of state fit into the abstract results in Section 2. Accordingly, we introduce in the present section first an appropriate functional frame work for parabolic evolution problems, following [SS]. In view of Theorem 2, we verify in particular the stability conditions (2.10), (2.11) for the nominal parabolic operator G_0, in the corresponding spaces X and Y and establish its mapping properties and bounded invertibility. We then present examples of optimal control problems, following [GK].

The functional setting of the nominal problem is next used to formulate results for its parametric version and, in particular, for precise statements of smallness of perturbations. Sufficient conditions are once more given to cast the parametric parabolic control problem into the abstract theory of Section 2, implying in particular analytic dependence of state and controls on the parameter vector σ. Sufficient conditions on the perturbations to ensure best N-term convergence rates will be identified.

4.1. Space–Time Variational Formulations of Parabolic State Equations. Denote by $\Omega_T := I \times \Omega$ with time interval $I := (0,T)$ the time–space cylinder for functions $f = f(t,x)$ depending on time t and space x. The parameter $T < \infty$ will always denote a finite time horizon. Let Y be a dense subspace of $H := L_2(\Omega)$ which is continuously embedded in $L_2(\Omega)$ and denote by Y' its topological dual. The associated dual form is denoted by $\langle\cdot,\cdot\rangle_{Y'\times Y}$ or, shortly $\langle\cdot,\cdot\rangle$. Later we will use $\langle\cdot,\cdot\rangle$ also for duality pairings between function spaces on the time-space cylinder Ω_T with the precise meaning clear from the context. We consider abstract parabolic problems as developed, e.g., in [L, Chapter III, pp. 100]. Specifically, we assume given for a.e. $t \in I$ and for $\sigma \in \mathcal{S}$ bilinear forms $a(\sigma, t; \cdot,\cdot) : Y \times Y \to \mathbb{R}$ so that $t \mapsto a_0(\sigma, t; \cdot,\cdot)$ is measurable on I and such that $a(\sigma, t; \cdot, \cdot)$ is continuous and elliptic on Y, uniformly in $t \in I$ and in $\sigma \in \mathcal{S}$: there exist constants $0 < \alpha_1 \leq \alpha_2 < \infty$ independent of t such that for a.e. $t \in I$ and for every $\sigma \in \mathcal{S}$

$$\begin{aligned} |a(\sigma, t; v, w)| &\leq \alpha_2 \|v\|_Y \|w\|_Y, && v, w \in Y, \\ a(\sigma, t; v, v) &\geq \alpha_1 \|v\|_Y^2, && v \in Y \ . \end{aligned} \tag{4.1}$$

By the Riesz representation theorem, there exists a one-parameter family of bounded, linear operators $A(\sigma, t) \in \mathcal{L}(Y, Y')$ such that

$$\forall \sigma \in \mathcal{S} : \quad \langle A(\sigma, t)v, w\rangle := a(\sigma, t; v, w), \qquad v, w \in Y \ . \tag{4.2}$$

Typically, $A(\sigma, t)$ will be a linear elliptic differential operator of second order on Ω and Y will denote a function space on Ω, such as, e.g., $Y = H_0^1(\Omega)$. We denote by $L_2(I; Z)$ the space of all functions $v = v(t, x)$ for which for a.e. $t \in I$ one has $v(t, \cdot) \in Z$. Instead of $L_2(I; Z)$, we will write this space as the (topological) tensor product of the two separable Hilbert spaces, $L_2(I) \otimes Z$, which, by [A, Theorem 12.6.1], can be identified.

For analytical purposes, it is convenient to interpret linear parabolic evolution equations as ordinary differential equations in an infinite-dimensional state space Y (see, e.g., [E]): given an initial condition $y_0 \in H$ and right-hand side $f \in L_2(I; Y')$, find $y(\sigma; \cdot)$ in some function space on Ω_T such that

$$\begin{aligned} \langle \tfrac{\partial y(\sigma; t, \cdot)}{\partial t}, v\rangle + \langle A(\sigma, t)\, y(\sigma; t, \cdot), v\rangle &= \langle f(t, \cdot), v\rangle && \text{for all } v \in Y \text{ and a.e. } t \in (0, T), \\ \langle y(0, \cdot), v\rangle &= \langle y_0, v\rangle && \text{for all } v \in H \ . \end{aligned} \tag{4.3}$$

In order to cast such parabolic equations of state into the abstract setting of Section 2 and as basis for the recently developed space-time adaptive, compressive discretizations of such equations of state, however, *space–time variational formulation* for (4.3) are required. One such formulation is based on the Bochner type *solution space*

$$\begin{aligned} \mathcal{X} &:= \{w \in L_2(I; Y) : \tfrac{\partial w(t, \cdot)}{\partial t} \in L_2(I; Y')\} = L_2(I; Y) \cap H^1(I; Y') \\ &= (L_2(I) \otimes Y) \cap \big(H^1(I) \otimes Y'\big) \end{aligned} \tag{4.4}$$

equipped with the graph norm

$$\|w\|_{\mathcal{X}}^2 := \|w\|_{L_2(I;Y)}^2 + \|\tfrac{\partial w(t, \cdot)}{\partial t}\|_{L_2(I;Y')}^2 \tag{4.5}$$

and the Bochner *space of test functions*

$$\mathcal{Y} := L_2(I; Y) \times H = (L_2(I) \otimes Y) \times H \tag{4.6}$$

equipped, for $v = (v_1, v_2) \in \mathcal{Y}$, with the norm

$$\|v\|_{\mathcal{Y}}^2 := \|v_1\|_{L_2(I;Y)}^2 + \|v_2\|_H^2 \tag{4.7}$$

Note that $v_1 = v_1(t, x)$ and $v_2 = v_2(x)$. (We remark in passing that the choices (4.4) of spaces incorporates the initial condition as *essential* condition in the space; other possible formulations allow for the initial condition as *natural* condition, see [ChSt] for details on such formulations which, in the present context of tracking type, high-dimensional parametric control problems, allow for completely analogous results).

Integration of (4.3) over $t \in I$ leads to the variational problem: given $f \in \mathcal{Y}'$, for every $\sigma \in \mathcal{S}$ find a function $y(\sigma) \in \mathcal{Y}$

$$b(\sigma; y(\sigma), v) = \langle f, v\rangle \qquad \text{for all } v = (v_1, v_2) \in \mathcal{Y}, \tag{4.8}$$

where the bilinear form $b(\sigma; \cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is defined by

$$b(\sigma; w, (v_1, v_2)) := \int_I \Big(\langle \tfrac{\partial w(t,\cdot)}{\partial t}, v_1(t, \cdot)\rangle + \langle A(\sigma, t)\, w(t, \cdot), v_1(t, \cdot)\rangle \Big)\, dt + \langle w(0, \cdot), v_2\rangle \tag{4.9}$$

and the right-hand side $\langle f + y_0, \cdot\rangle : \mathcal{Y} \to \mathbb{R}$ by

$$\langle f, v_1\rangle + \langle y_0, v_2\rangle := \int_I \langle f(t, \cdot), v_1(t, \cdot)\rangle\, dt + \langle y_0, v_2\rangle \tag{4.10}$$

for $v = (v_1, v_2) \in \mathcal{Y}$. It is well-known (see, e.g. [DL, Chapter XVIII, Sect. 3]) that the parametric operator family $\{B(\sigma) : \sigma \in \mathcal{S}\}$ defined by the bilinear form $b(\sigma; \cdot, \cdot)$ in (4.9) is a family of isomorphisms from $\mathcal{X}$ to $\mathcal{Y}'$. In [SS], detailed bounds on the norms of the operator and its inverse were established. To prepare the ensuing formulation and regularity results on the parametric parabolic optimal control problem, we next formulate the corresponding result for the state equation (4.8). This result is again a special case of the abstract results, Theorem 4 and Theorem 7. Alternatively, it could be inferred from the abstract theory of parabolic evolution equations in [PS], subject to a requirement of continuity of $A(\sigma, t)$ with respect to $t \in [0, T]$, uniformly with respect to $\sigma \in \mathcal{S}$.

Recall that we have assured for elliptic diffusion problems that the parametric family $\{A(\sigma, t) \in \mathcal{L}(Y, Y') : \sigma \in \mathcal{S}, t \in I\}$ is $\mathfrak{p}$ regular for arbitrary $0 < \mathfrak{p} \leq 1$ since it is affine according to (3.30). In order to cover also more general cases, we assume again for the following result that $\{A(\sigma, t) \in \mathcal{L}(Y, Y') : \sigma \in \mathcal{S}, t \in I\}$ is $\mathfrak{p}$-regular for some $0 < \mathfrak{p} \leq 1$.

THEOREM 21. *For every $\sigma \in \mathcal{S}$, the parabolic evolution operator $B(\sigma) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ defined by $\langle B(\sigma)w, v\rangle := b(\sigma; w, v)$ for $w \in \mathcal{X}$ and for $v \in \mathcal{Y}$ with the parametric bilinear form $b(\sigma; \cdot, \cdot)$ from* (4.9) *and with the choice of spaces $\mathcal{X}$, $\mathcal{Y}$ as in* (4.4) *and* (4.6) *is boundedly invertible: there exist constants $0 < \beta_1 \leq \beta_2 < \infty$ such that*

$$\sup_{\sigma\in\mathcal{S}} \|B(\sigma)\|_{\mathcal{X}\to\mathcal{Y}'} \leq \beta_2 \quad \text{ and } \quad \sup_{\sigma\in\mathcal{S}} \|B(\sigma)^{-1}\|_{\mathcal{Y}'\to\mathcal{X}} \leq \frac{1}{\beta_1}. \tag{4.11}$$

Moreover, the parametric operator family $\{B(\sigma) : \sigma \in \mathcal{S}\}$ satisfies Assumption 1 with the same regularity parameter $\mathfrak{p}$. In particular, the parametric family $y(\sigma)$ in (4.8) *of states satisfies the a-priori estimate*

$$\forall \nu \in \mathfrak{F} : \quad \sup_{\sigma\in\mathbb{S}} \|(\partial_\sigma^\nu y)(\sigma)\|_{\mathcal{X}} \leq C_0 \|f\|_{\mathcal{Y}'} |\nu|! \tilde{b}^\nu\ , \tag{4.12}$$

and admits a Legendre expansion

$$y(\sigma) = \sum_{\nu\in\mathfrak{F}} y_\nu(\sigma) L_\nu(\sigma)\ , \quad y_\nu = \int_{\sigma\in\mathcal{S}} y(\sigma) L_\nu(\sigma) \rho(d\sigma) \tag{4.13}$$

which converges in $L^2(\mathcal{S}, \rho; \mathcal{X})$. The sequences $(\|y_\nu\|_{\mathcal{X}})_{\nu\in\mathfrak{F}} \in \ell^{\mathfrak{p}}(\mathfrak{F})$ and their best N-term truncated Legendre expansions converge at rate $N^{-(1/\mathfrak{p}-1/2)}$ in $L^2(\mathcal{S}, \rho; \mathcal{X})$.

Proof. As proved in [SS], for every $\sigma \in \mathcal{S}$ the continuity constant β_2 and the inf–sup condition constant β_1 for $b(\sigma;\cdot,\cdot)$ are independent of $\sigma \in \mathcal{S}$ and satisfy

$$\beta_1 \geq \frac{\min(\alpha_1\alpha_2^{-2},\alpha_1)}{\sqrt{2\max(\alpha_1^{-2},1)+\varrho^2}}, \qquad \beta_2 \leq \sqrt{2\max(1,\alpha_2^2)+\varrho^2}, \tag{4.14}$$

where α_1,α_2 are the constants from (4.1) bounding $A(\sigma,t)$ and ϱ is defined as

$$\varrho := \sup_{0\not\equiv w\in\mathcal{Y}} \frac{\|w(0,\cdot)\|_H}{\|w\|_{\mathcal{Y}}}. \tag{4.15}$$

We like to recall from [DL, E] that $\mathcal{Y}$ is continuously embedded in $\mathcal{C}^0(I;H)$ so that the pointwise in time initial condition in (4.3) is well-defined. From this it follows that the constant ϱ is bounded uniformly for the choice of $\mathcal{Y} \hookrightarrow H$. □

In the sequel, we will require the dual operator $B(\sigma)^* : \mathcal{Y} \to \mathcal{X}'$ of $B(\sigma)$ which is defined formally by

$$\forall\sigma\in\mathcal{S}: \quad \langle B(\sigma)w,v\rangle =: \langle w, B(\sigma)^*v\rangle\ . \tag{4.16}$$

From the definition of the bilinear form (4.9) on $\mathcal{X}\times\mathcal{Y}$, it follows by integration by parts for the first term with respect to time and using the adjoint $A(\sigma,t)^*$ with respect to the duality pairing $Y'\times Y$ that

$$\begin{aligned} b(\sigma;w,(v_1,v_2)) \;=\; & \int_I \Big(\langle w(t,\cdot), \tfrac{\partial v_1(t,\cdot)}{\partial t}\rangle + \langle w(t,\cdot), A(\sigma,t)^* v_1(t,\cdot)\rangle\Big)\, dt \\ & + \;\langle w(0,\cdot), v_2\rangle + \langle w(t,\cdot), v_1(t,\cdot)\rangle|_0^T \\ =:\; & \langle w, B(\sigma)^* v\rangle. \end{aligned} \tag{4.17}$$

Note that the first term of the right-hand side which involves $\frac{\partial}{\partial t}v_1(t,\cdot)$ is still well-defined with respect to t as an element of $\mathcal{Y}'$ on account of $w\in\mathcal{Y}$.

So far, we considered only the parabolic state equation and proved analyticity and polynomial approximation rates.

We now turn to perturbed, parametric state equations resulting from parametric uncertainty in the spatial operator $A(\sigma,t)$, and present in particular sufficient conditions on the perturbations of $A_0(t)$ in order for the perturbed state equation to fit into the general Assumption 2 and Theorem 2.

4.2. Tracking-type control problem constrained by a parametric, parabolic PDE. Recalling the situation from [GK], we wish to minimize, for some given target state y_* and fixed end time $T>0$, the quadratic functional

$$J(y,u) := \tfrac{\omega_1}{2}\|y-y_*\|^2_{L_2(I;O)} + \tfrac{\omega_2}{2}\|y(T,\cdot)-y_*(T,\cdot)\|^2_O + \tfrac{\omega_3}{2}\|u\|^2_{L_2(I;U)} \tag{4.18}$$

over the state $y(\sigma)=y(\sigma;t,x)$ and over the control $u(\sigma)=u(\sigma;t,x)$ subject to

$$B(\sigma)y(\sigma) = \mathfrak{E}u(\sigma) + \begin{pmatrix} f \\ y_0 \end{pmatrix} \qquad \text{in } \mathcal{Y}', \tag{4.19}$$

where $B(\sigma)$ denotes the parametric, parabolic evolution operator defined by Theorem 21 and where $f\in\mathcal{Y}'$ is given by (4.10). In (4.18), the real weight parameters $\omega_1,\omega_2\geq 0$ are such that $\omega_1+\omega_2>0$ and $\omega_3>0$. The space O by which the integral over

Ω in the first two terms in (4.18) is indexed is to satisfy $O \supseteq Y$ with continuous embedding. Although there is in the wavelet framework great flexibility in choosing even fractional Sobolev spaces for O, for better readibility, we pick here $O = Y$. Moreover, in a general case we suppose that the operator $\mathfrak{E}$ is a linear operator $\mathfrak{E} : U \to \mathcal{Y}'$ extending $\int_I \langle u(t,\cdot), v_1(t,\cdot)\rangle\, dt$ trivially, i.e., $\mathfrak{E} \equiv (I, 0)^\top$. For ease of presentation in the current setting, we choose again $U = Y'$ similar to the stationary case in Section 3.1.

The tracking type control problem consists in minimizing the functional (4.18) subject to the parametric parabolic equation of state (4.19). We recall that the *Riesz operator* $R_Y : Y \to Y'$ defined by

$$(v, z)_Y =: \langle v, R_Y z\rangle, \quad v, z \in Y, \tag{4.20}$$

maps Y boundedly invertibly onto its dual Y'. Since here $R_U = R_Y^{-1}$ as in Section 3.1, we write $R = R_Y$.

Analogously to the derivation of the system (EE) in Section 3.1, we can derive the first order necessary conditions consisting of the *primal system* together with the *costate* or *adjoint equations* and the *design equation*. For a unification of notation, it will be useful to define

$$y_1(\sigma) := y(\sigma), \qquad y_2(\sigma) := y(\sigma; 0)$$

and, since the adjoint state also requires the state to be evaluated at the finite end point (sometimes also denoted as *finite horizon*) T, $y_3(\sigma) := y(\sigma; T)$. Then the necessary conditions for optimality read:
find the solution tuple $(y_1(\sigma), y_2(\sigma), y_3(\sigma), p_1(\sigma), p_2(\sigma), u(\sigma)) \in \mathcal{X}\times Y\times Y\times\mathcal{X}\times Y\times\mathcal{Y}'$ as

$$\begin{aligned}
B_1(\sigma)\, y_1(\sigma) &= u(\sigma) + f \\
B_2(\sigma) y_2(\sigma) &= y_0 \\
B_1(\sigma)^* p_1(\sigma) + \omega_1 R_Y y_1(\sigma) &= \omega_1 R_Y y_* \\
B_2(\sigma)^* p_2(\sigma) + \omega_2 R_Y y_3(\sigma) &= \omega_2 R_Y y_*(T) \\
\omega_3 u(\sigma) &= R_Y p_1(\sigma)\ .
\end{aligned} \tag{4.21}$$

Here $B_1(\sigma), B_2(\sigma)$ are the linear operators defined by the first and second dual forms in (4.9), respectively, with 'dual' $B_1(\sigma)^*, B_2(\sigma)^*$ defined according to (4.17). Note that the appearance of the Lagrange multipliers $p_1(\sigma), p_2(\sigma)$ is caused by appending the parabolic constraints (4.19) to the functional (4.18). Thus, the variable $p_1(\sigma)$ is the adjoint state $p_1(\sigma) = p(\sigma; t, x)$, and $p_2(\sigma)$ may be interpreted as evaluating p at the end point T, i.e., $p_2(\sigma) = p(\sigma; T, x)$. For presentation purposes, we also define $p_3(\sigma) = p(\sigma; 0, x)$. Eliminating $u(\sigma) = \omega_3^{-1} R_Y p_1(\sigma)$ from the design equation and abbreviating

$$\hat{y}_* := R_Y y_* \quad \text{and} \quad \hat{y}_*(T) := R_Y y_*(T)\ ,$$

and

$$\widehat{y(\sigma)} = (y_1(\sigma), y_2(\sigma), y_3(\sigma)), \quad \widehat{p(\sigma)}) = (p_1(\sigma), p_2(\sigma), p_3(\sigma))$$

we arrive at the coupled system

$$G(\sigma)\begin{pmatrix}\widehat{y(\sigma)}\\ \widehat{p(\sigma)}\end{pmatrix} := \begin{pmatrix} \widehat{B(\sigma)} & \mathrm{diag}(-\frac{1}{\omega_3}R_Y,0,0) \\ \begin{pmatrix}\frac{1}{\omega_1}R_Y & 0 & 0\\ 0 & 0 & \frac{1}{\omega_2}R_Y\\ 0 & 0 & 0\end{pmatrix} & \widehat{B(\sigma)}^* \end{pmatrix}\begin{pmatrix}\widehat{y(\sigma)}\\ \widehat{p(\sigma)}\end{pmatrix} \tag{4.22}$$

$$= \begin{pmatrix} f\\ y_0\\ 0\\ \omega_1\hat{y}_*\\ \omega_2\hat{y}_*(T)\\ 0\end{pmatrix} =: g$$

where $\widehat{B(\sigma)} := \mathrm{diag}(B_1(\sigma), B_2(\sigma), 0)$. For the final result, we assume again that the parametric family of spatial operators $\{A(\sigma,t) \in \mathcal{L}(Y,Y') : \sigma \in \mathcal{S}\}$ satisfies Assumption 1 for some regularization parameter $0 < \mathfrak{p} \leq 1$. Applying the arguments as for the proof of Theorem 21, Corollary 20 and Theorem 7, we arrive at

THEOREM 22. *Assume that for every $t \in [0,T]$ the parametric family of spatial operators $\{A(\sigma,t) \in \mathcal{L}(Y,Y') : \sigma \in \mathcal{S}\}$ satisfy Assumption 1. Then*

(i) for every $\sigma \in \mathcal{S}$, the tracking type control problem (4.21) *can be written as a parametric saddle point operator equation $G(\sigma)(\widehat{y(\sigma)}, \widehat{p(\sigma)}) = g$ for the solution tuple $(\widehat{y(\sigma)}, \widehat{p(\sigma)}) \in \boldsymbol{\mathcal{X}}$ with $G(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}')$ where the space $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}}$ is given by*

$$\boldsymbol{\mathcal{X}} = \mathcal{X} \times Y \times Y. \tag{4.23}$$

(ii) Moreover, for $\omega_1 + \omega_2 > 0$, $\omega_3 > 0$, the parametric saddle point operator $\mathbf{G}(\sigma) \in \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}})$ in (4.22) *is boundedly invertible for all $\sigma \in \mathcal{S}$ and satisfies Assumption 1 with the same regularity parameter $\mathfrak{p}$.*

(iii) The parametric family of state-costate pairs $\mathcal{S} \ni \sigma \mapsto \begin{pmatrix}\widehat{y(\sigma)}\\ \widehat{p(\sigma)}\end{pmatrix} \in L^2(\mathcal{S}, \rho; \boldsymbol{\mathcal{X}})$ depends analytically on $\sigma \in \mathcal{S}$.

(iv) The parametric family of state-costate pairs admits a concurrent *Legendre expansion*

$$\begin{pmatrix}\widehat{y(\sigma)}\\ \widehat{p(\sigma)}\end{pmatrix} = \sum_{\nu\in\mathfrak{F}} L_\nu(\sigma)\begin{pmatrix}y_\nu\\ p_\nu\end{pmatrix}, \quad \begin{pmatrix}y_\nu\\ p_\nu\end{pmatrix} \in \boldsymbol{\mathcal{X}}. \tag{4.24}$$

(v) Furthermore, the parametric Legendre expansion is sparse, *i.e., the coefficient sequence in* (4.24) *is $\mathfrak{p}$-summable,*

$$\left(\left\|\begin{pmatrix}y_\nu\\ p_\nu\end{pmatrix}\right\|_{\boldsymbol{\mathcal{X}}}\right)_{\nu\in\mathfrak{F}} \in \ell^{\mathfrak{p}}(\mathfrak{F})$$

for the same value of $\mathfrak{p}$,

(vi) For every $N \in \mathbb{N}$, there exists an index set $\Lambda \subset \mathfrak{F}$ of cardinality not exceeding N such that the N-term truncated Legendre expansions

$$\begin{pmatrix}y_N(\sigma)\\ p_N(\sigma)\end{pmatrix} := \sum_{\nu\in\Lambda} L_\nu(\sigma)\begin{pmatrix}y_\nu\\ p_\nu\end{pmatrix}, \quad \begin{pmatrix}y_\nu\\ p_\nu\end{pmatrix} \in \boldsymbol{\mathcal{X}},$$

approximates simultaneously *the state and the control on the entire parameter domain* $\mathcal{S}$ *at rate* $N^{-(1/\mathfrak{p}-1/2)}$ *in* $L^2(\mathcal{S},\rho;\boldsymbol{\mathcal{X}})$.

We emphasize again that the *same index set* Λ can be used for *concurrent approximation of both variables* (y,p) (see, eg., Corollary 20) and, on account of the relation between the control u and p (4.21), also for u. This is caused by formulating the energy equations as a saddle point problem and applying the abstract theory from Section 2.4. This fact will facilitate the actual computation of this index set.

5. Conclusion. We have proved, for control problems constrained by linear elliptic and parabolic PDEs which depend on possibly countably infinitely many parameters, analytic parameter dependence of the state, co-state and of the control. The parameter dependence was allowed to be more general than affine. The particular case of affine dependence arises, for example, in state equations with random coefficients which are parametrized in terms of Karhunen-Loève expansions as in [ST]. We have quantified the analytic dependence of (co)state and control. Specifically, we established that these quantities allow expansions in terms of tensorized polynomial chaos type bases which are sparse, their sparsity being quantified in terms of $\mathfrak{p}$-summability of the coefficient sequences. This sparsity result which may be viewed as an a-priori estimate with respect to the parameters is the analytical foundation for the development of sparse tensor discretizations of these problems. After having established in Theorem 22 the existence of index sets Λ for which gpc expansions of state and control attain rates of best N-term approximation, following the ideas in [CCDS, G], adaptive Galerkin approximations of (co)state and control on the entire (possibly infinite-dimensional) parameter space which realize optimal approximation rates can be computed by greedy algorithms; for implementational aspects for elliptic forward problems, we refer to [EGSZ]. The adaptive Galerkin discretization algorithms developed there for the stochastic forward problem can be combined with finite element or wavelet methods for control problems as presented in [BoKu, BoSch, Bu1, DK, EG, GK, GY, K, MV, Pa, SSZ]. Such sparse tensor Galerkin approximations, combined with appropriate discretizations in space and time, are the subject of the forthcoming manuscript [KS].

The results on quantitative parameter dependence of state and costate obtained in Section 2.3 are the basis for the use of Quasi-Monte-Carlo (QMC) quadrature schemes developed in [KSS] for the case of elliptic PDEs with random coefficients. These are somewhat easier to implement but currently are proven to realize the optimal N-term rates for the range $\frac{2}{3} < \mathfrak{p} \le 1$. Higher order QMC methods currently under development will appeal also to the regularity results in Section 2.3 of the present paper.

REFERENCES

[A] J.P. Aubin, Applied Functional Analysis, 2nd ed., Wiley, 2000.

[BNT] I. Babuska, F. Nobile, and R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, SIAM J. Numer. Anal. 45 (2007), 1005–1034.

[BTZ] I. Babuska, R. Tempone, and G. E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, SIAM J. Numer. Anal., 42 (2004), 800–825.

[BoKu] A. Borzi and K. Kunisch, A multigrid scheme for elliptic constrained optimal control problems, Comp. Optim. and Appl., 31 (2005), 309–333.

[BoSch] A. Borzi and V. Schulz, Multigrid methods for PDE optimization, SIAM Rev., 51 (2009), 361–395.

[BF] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer Verlag, Berlin, 1991.

[Bu1] C. Burstedde, Fast Optimised Wavelet Methods for Control Problems Constrained by Elliptic PDEs, Dissertation, Universität Bonn, 2005, `http://hss.ulb.uni-bonn.de/2005/0682/0682.htm`.

[Bu2] C. Burstedde, On the numerical evaluation of fractional Sobolev norms, Commun. Pur. Appl. Analysis 6(3) (2007), 587 - 605.

[CR] E. Casas and J.-P. Raymond, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, SIAM J. Control Optim. 45(5) (2006), 1586–1611.

[ChSt] N. Chegini and R. Stevenson, Adaptive wavelet schemes for parabolic problems: sparse matrices and numerical results, SIAM J. Numer. Anal. 49 (2011), 182-212.

[CCDS] A. Chkifa, A. Cohen, R. DeVore and Ch. Schwab, Sparse adaptive Taylor approximation algorithms for parametric and stochastic elliptic PDEs, Report 2011-44, Seminar for Applied Mathematics, ETH Zürich (to appear in M2AN 2013).

[CDS1] A. Cohen, R. DeVore and Ch. Schwab, Convergence rates of best N-term Galerkin approximation for a class of elliptic sPDEs, Found. Comput. Math. 10(6) (2010), 615–646.

[CDS2] A. Cohen, R. DeVore and Ch. Schwab, Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs, Analysis and Applications 9(1) (2011), 1–37.

[DK] W. Dahmen and A. Kunoth, Adaptive wavelet methods for linear–quadratic elliptic control problems: Convergence rates, SIAM J. Contr. Optim. 43(5) (2005), 1640–1675.

[DL] R. Dautray and J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, Vol.5: Evolution Problems, Springer, 2000.

[DFT] A. Debussche, M. Fuhrman, and G. Tessitore, Optimal control of a stochastic heat equation with boundary-noise and boundary-control, ESAIM Control Optim. Calc. Var., 13 (2007), 178–205.

[DeV] R. DeVore, Nonlinear approximation, Acta Numerica 7 (1998), 51–150.

[DeVK] R. DeVore, A. Kunoth (Eds.), Multiscale, Nonlinear and Adaptive Approximation, Springer, 2009.

[EGSZ] M. Eigel, C.J. Gittelson, Ch. Schwab and E. Zander, Adaptive Stochastic Galerkin FEM, Report 2013-01, Seminar for Applied Mathematics, ETH Zürich (in review).

[EG] M. Engel, M. Griebel, A multigrid method for constrained optimal control problems. J. Comput. Appl. Math. 235 (15) (2011), 4368–4388.

[E] L.C. Evans, Partial Differential Equations, AMS, Providence, RI, 1998.

[GS] R. Ghanem and P.D. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer Verlag, 1991, reissued by Dover Publications, 2004.

[G] C.J. Gittelson, Adaptive stochastic Galerkin methods: Beyond the elliptic case. Report 2011-12, Seminar for Applied Mathematics, ETH Zürich (in review).

[GY] W. Gong and N. Yan, Mixed finite element method for Dirichlet boundary control problem governed by elliptic PDEs, SIAM J. Control Optim. 49(3) (2011), 984–1014.

[GW] A. Griewank and A. Walther, Revolve: An implementation of checkpointing for the reverse or adjoint mode of computational differentiation, ACM Trans. Math. Software **26** (2000), 19-45.

[GK] M.D. Gunzburger and A. Kunoth, Space-time adaptive wavelet methods for control problems constrained by parabolic evolution equations, SIAM J. Contr. Optim. 49(3) (2011) 1150–1170.

[GLL] M.D. Gunzburger, H.–C. Lee and J. Lee, Error estimates of stochastic optimal Neumann boundary control problems, SIAM J. Numer. Anal. 49(4) (2011), 1532-1552.

[Ha] W. Hackbusch, Fast solution of elliptic control problems, J. Optim. Theory Appl. 31 (4) (1980), 565 - 581.

[HaSch] H. Harbrecht and R. Schneider, Rapid solution of boundary integral equations by wavelet Galerkin schemes, in: [DeV], 249 - 294.

[HLM] L.S. Hou, J. Lee, and H. Manouzi, Finite element approximations of stochastic optimal control problems constrained by stochastic elliptic PDEs, J. Math. Anal. Appl. 384(1) (2011), 87-103.

[KX] G.E. Karniadakis and D.B. Xiu, The Wiener-Askey polynomial chaos for stochastic differential equations, SIAM J. Sci. Comput. 24 (2002), 619–644.

[KV] K. Kunisch and B. Vexler, Constrained Dirichlet boundary control in L^2 for a class of evolution equations, SIAM J. Contr. Optim. 46(5) (2007), 1726–1753.

[K] A. Kunoth, Adaptive wavelet schemes for an elliptic control problem with Dirichlet boundary control, Numer. Algor. 39(1-3) (2005), 199-220.

[KS] A. Kunoth, and Ch. Schwab, Sparse adaptive tensor Galerkin approximations of stochastic

PDE-constrained control problems (in preparation 2013).

[KSS] F.Y. Kuo, Ch. Schwab and I.H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients, to appear in SIAM. J. Numer. Anal. 2013.

[L] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, Berlin, 1971.

[MV] D. Meidner and B. Vexler, Adaptive space-time finite element methods for parabolic optimization problems SIAM J. Contr. Optim. 46(1), (2007), 116–142.

[Ng] H. Nguyen, Finite element wavelets for solving partial differential equations. PhD thesis, Utrecht University, 2005.

[Pa] R. Pabel, Wavelet Methods for PDE Constrained Control Problems with Dirichlet Boundary Control, Shaker Publisher, 2007, doi:10.2370/236_232.

[RW] H. Rauhut and R. Ward. Sparse Legendre expansions via ℓ_1-minimization. J. Approx. Theory, 164(5), (2012), 517–533.

[PS] J. Prüss and R. Schnaubelt, Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time, J. Math. Anal. Appl. 256 (2001), 405-430.

[SSZ] J. Schöberl, R. Simon, W. Zulehner, A robust multigrid method for elliptic optimal control problems. SIAM J. Numer. Anal. 49 (4) (2011), 1482–1503.

[Sch] W. Schoutens, Stochastic Processes and Orthogonal Polynomials, Lecture Notes in Statistics, Vol. 146, Springer, 2000.

[SG] Ch. Schwab and C.J. Gittelson, Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs, Acta Numerica 20 (2011), Cambridge University Press, Cambridge, UK.

[SS] Ch. Schwab and R. Stevenson, Space–time adaptive wavelet methods for parabolic evolution equations, Math. Comp. 78, (2009) 1293–1318.

[ST] Ch. Schwab and R.A. Todor, Karhúnen-Loève approximation of random fields by generalized fast multipole methods, Journal of Computational Physics 217 (2006), 100-122.

[W] N. Wiener, The homogeneous chaos, Am. J. Math. 60 (1938), 897–936.